INVESTIGATING THE USE OF MODELS TO DEVELOP GRADE 8 LEARNERS' CONCEPTUAL UNDERSTANDING OF AND PROCEDURAL FLUENCY WITH FRACTIONS

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ABSTRACT

Both my teaching experience and literature of this research study strongly suggested that fractions are difficult to teach and learn across the globe generally, and Namibia in particular. One of the identified contributing factors was teaching fractions by focusing on procedures and not the conceptual understanding. Therefore, this research project developed and implemented an intervention in order to experiment and suggest an alternative teaching approach of fractions using models. The purpose of this research was to: "Investigate the use of models to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions". This investigation had three areas of focus. Firstly, the study investigated the nature of learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention, by means of administering a pre-test and pre-interview and analysing learners' responses. Secondly, the study investigated the changes in learners' conceptual understanding of and procedural fluency with fractions after the teaching intervention, by means of administering a post-test, post-interviews and recall interviews, and analysing learners' responses. Thirdly, this study investigated the possible influence of the teaching intervention on the changes in learners' conceptual understanding of and procedural fluency with fractions by analysing the lesson videos and learners' worksheets, and describe their critical interaction.

This study was conducted at a multicultural urban secondary school located in the Oshikoto Region, Namibia. The sample consisted of 12 Grade 8 mathematics learners whose age ranged from 13-16 years old. A purposive sampling method was employed to select both the research site and participants. This research is framed as a case study, and is grounded within the interpretive paradigm and qualitative research.

This research revealed that these learners displayed conceptual and procedural difficulties in their engagement with fraction models and fraction symbols, before the teaching intervention. Conceptually, the study found that these learners read fractions using inappropriate names; and learners did not identify the whole unit in the models and therefore identified fractions represented by the fraction models using different forms of inappropriate fraction symbols. Procedurally, the study found that these learners compared and ordered fractions inappropriately using the sizes of the numerators and denominators separately; and learners

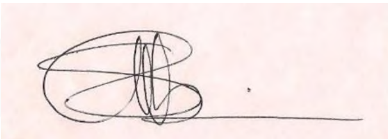
used the lowest common denominator method inappropriately for adding fractions with different denominators.

The research also suggested conceptual and procedural changes in learners' conceptual understanding of and procedural fluency with fractions and that the intervention seemed to help learners to engage better with fraction models and fraction symbols. Conceptually, the findings suggested that the intervention using area models and number lines, seemed to help these learners to read fractions using appropriate names; to identify the whole unit in the fraction models and to develop a sense of the size of fractions in relation to one whole unit. Procedurally, the learners compared and ordered fractions appropriately using either equal fraction bars, equal number lines, benchmarking or rules for comparing and ordering fractions with the same numerator or denominator; and learners used equal fraction bars to visually represent the lowest common denominator method and to recognise that only equally sized units can be counted together.

This research identified four factors as possible influences of the teaching intervention. These factors are namely: identifying both fraction symbols and appropriate fraction names to see fractions as relational numbers; prompting to partition whole units of the fraction models and graphically illustrating fraction symbols to identify the whole unit in the fraction models and to develop a sense of the size of fractions in relation to one whole unit; graphically illustrating fraction symbols using the models to use equal fraction bars and number lines, benchmarking and rules for comparing; and graphically illustrating fraction denominations using equal fraction bars to recognise that only equally sized units can be counted together. This research strongly suggests that the effective use of models has the potential to develop learners' conceptual understanding of and procedural fluency with fractions in a number of ways.

DECLARATION OF ORIGINALITY

I, Simon Albin, student number: 13A6312, declare that this thesis *"Investigating the use of models to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions"* is my own work written in my own words. Where I have drawn on the words and ideas of others, these have been fully acknowledged according to the Rhodes University Education Department Referencing Guide. Likewise, this thesis has not been submitted in any form for another qualification or any assessment to another University or institution.



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(Date)

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DEDICATION

On behalf of my students, family and friends, I dedicate this humble work to the memory of my beloved grandmother *Ester Nandago "Nambunga" Nangolo* for nursing me with courage to work hard and never give up. I had promised to make you very proud and I hope that this monumental academic achievement has partially fulfilled the promise.

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

This research seeks to answer the following question: *how can we help learners to understand fractions better, particularly in Namibian schools?* Firstly, this chapter welcomes the reader to the research project by describing the context of the study and the motivation for the research. Secondly, it describes the rationale for the study, research goals, research questions, and significance of the study. Finally, it presents the outline of the structure of this dissertation.

1.2 BACKGROUND AND CONTEXT OF THE STUDY

In many countries including Namibia, annual examinations in mathematics are used to determine if learners can proceed to the next grade (Schoenfeld, 2007). According to the promotion policy requirements for schools in Namibia, every learner in Grade 8 and 9 should "obtain E grades or better in six subjects including English and Mathematics or F grades or better in the remaining three subjects" (Sichombe, Nambira, Tjipueja, & Kapenda, 2011, p. 20). As a Grade 8 mathematics teacher for about four years in the northern part of Namibia, I found myself needing to ensure that learners do well enough in mathematics to get promoted. Currently there are nine subjects in Grade 8 and mathematics is one of the compulsory promotion subjects to the next grade. During my teaching career, I have witnessed many learners repeating Grade 8 because they could not obtain an E grade or better, in Mathematics.

The teaching of mathematics in Namibian schools is offered in Grade 1-12 with Preparatory Mathematics being the first course of mathematics taken by learners in the pre-primary schools (Namibia. Ministry of Education, 2010, p. 12). Mathematics is a highly valued part of the Namibian Curriculum of Basic Education in order to develop numerical skills which are deemed to be indispensable tools for the development of science, technology and commerce (Namibia. Ministry of Education, 2010, p. 12). According to Namibia's Vision 2030, learners are expected to be fully numerate and able to create "logical models for understanding, and ...

to think in terms of relationships of quantity, size, shape and space, and computation" (Namibia. Ministry of Education, 2010, p. 12). Additionally, mathematics learners are required to understand and "use mathematical language confidently and effectively as a means of communication" (Namibia. Ministry of Education, 2010, p. 12). However, this dream may take longer than expected due to the present status quo of the difficulties of teaching and learning mathematics in schools.

In my experience, fractions is one of the most difficult topics in mathematics, both to teach and learn. Learners in Grade 8 are expected to be able to compare and order fractions as well as to perform the four basic operations on fractions (Namibia. Ministry of Education, 2010, p. 9). Unfortunately, many learners in this grade find it difficult to compare, order, and add fractions. I have also observed that many teachers get overwhelmed with anxiety whenever it is time to teach fractions. In her study, Vatilifa (2012) found that many student teachers have limited understanding of fractions (p. iii). She also found that many student teachers revert to the old practices of teaching fractions, even when they attempt to use new approaches to teach fractions meaningfully. According to her, the teaching of fractions is currently taught only through symbolic representations of abstract fraction concepts, and an incorrect use of "terminologies such as '1 over 4' or '1 out of 4' instead of a 'quarter' or 'one-fourth'" is common (Vatilifa, 2012, p. iii). She further reveals that some student teachers interpret a fraction as a pair of two different whole numbers that can be broken apart (Vatilifa, 2012, p. iii). I strongly concur with the findings of Vatilifa, having experienced the same challenges in teaching fractions.

In June 2014, I was part of a mathematics education students' class who attended the Rhodes University Bachelor of Education Honours programme of Namibian students in Okahandja. I was introduced to Singapore mathematics and to VITAL maths video clips relating to the teaching of fractions. I was amazed to see how models were used to teach fractions in a fun, interactive way and to increase learners' opportunities to develop numerical skills and creativity. I realised that such fun has been missing in my class when I taught fractions. It was for this reason that my curiosity inspired me to research the use and impact of using models to teach fractions.

In my experience, learners are not afforded opportunities to see fractions as quantities with specific magnitudes. Instead, they see them as abstract symbols that they have to memorise. I strongly believe that this makes it difficult for them to compare and order fractions correctly. Similarly, learners are only taught procedures for fraction arithmetic. The teaching of procedures for fraction addition, does not help many learners to find the correct sum. Brijlall, Maharaj and Molebale (2011) explain the difficulty: learners tend to either use procedures of whole number arithmetic or perform a combination of incorrect and correct procedures of fraction arithmetic. For instance, learners use multiplication procedures to work out the sum of two fractions, yielding an incorrect calculation, for example: 1/3+3/4=3/12. I am of the belief that a better approach to teaching fractions needs to be found and thus I have designed and implemented a teaching intervention programme, to investigate the use of models to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions.

This study is founded on the following belief or hypothesis, explained below (Kilpatrick, Swafford, & Findell, 2001):

> Mathematics requires representations (models) … [as representations help to] clarify mathematical ideas in ways that support reasoning and build understanding. These representations also support the development of efficient algorithms for the basic operations. (p. 95)

In the domain of fractions, a significant indicator of conceptual understanding is being able to represent mathematical ideas in different ways and "knowing how different representations can be useful for different purposes" (Kilpatrick et al., 2001, p. 119). It involves being able to see how various representations of fractions are connected with each other, how similar or different they are to each other (*ibid.*). Learners whose conceptual understanding of fractions is well developed often have less to master and "avoid many critical errors in solving problems, particularly errors of magnitudes" (*ibid.*, p. 120). On the other hand, procedural fluency of fractions includes knowing when and how to calculate the lowest common denominator to add fractions appropriately (Gabriel et al., 2012). Regrettably, Gabriel et al. (2012) note that in many instances, most learners appear to apply procedures or perform calculations without knowing the underlying concepts. Yet, there is hope for teaching fractions meaningfully since results of mathematical learning studies strongly indicate that conceptual knowledge can interactively and iteratively facilitate the generation and adoption of procedural knowledge (Gabriel et al., 2012, p. 138). For instance, the findings of a study by Pantziara and Philippou (2011) with sixth graders revealed that "students who rely only on procedural knowledge have lower performance on fractions than students who also gain conceptual knowledge" (Pantziara & Philippou, 2012, p. 79).

In their study, Pantziara and Philippou (2012) suggest the use of representations and the alternation between representations as "substantial elements for the development of students' conceptual knowledge" (p. 79). In this research study, two fraction models, namely the part-whole area model and the number line model were used to teach fractions conceptually and procedurally during the intervention. These models were used iteratively with fraction symbols to help learners to: (a) quantify fractions; (b) to meaningfully compare and order fractions and; (c) to meaningfully add two fractions.

Harvey (2011) indicates that "there is a range of models commonly used to support fraction instruction" (p. 334) such as sets of discrete objects, number lines (linear model), double number lines and area models, such as circles and rectangles (Watanabe, 2002; Harvey, 2011). According to Watanabe (2002), the typical common fraction models used in elementary and middle school mathematics textbooks, are linear models, area models and discrete models. The use of visual models in mathematics, especially number lines and area models to represent fractions, is recommended by many, including the National Mathematics Advisory Panel's Critical Foundation for Algebra (Council of Chief State School Officers, 2010), Texas Higher Education Coordinating Board (2009) and NCTM's Principles and Standards for School Mathematics (2000). Gray (2014) strongly indicates that the use of fraction models in the middle grades is considered a key to learners' success, to master or "conceptually anchor the algorithms used to work with fractions" (p. 7) and for learners to make connections among models, which eventually deepen their conceptual understanding (Gray, 2014). Similarly, Van de Walle, Karp, Bay-Williams, Wray and Rigelman (2013) also state that the effective use of fraction models can help learners to "clarify ideas that are often confused in a purely symbolic model" (p. 342). The same paper also stipulates that at times the use of two different and appropriate models coupled with asking learners to make connections between the models is an effective way of broadening and deepening both teachers' and learners' understanding of fractions (Van de Walle et al., 2013). The use of different fraction models is recommended because different models are known to offer different learning opportunities. For instance, Cramer, Wyberg and Leavitt (2008) found that the fraction circle (area) model is the most powerful part-whole model for building mental images of the relative size of fractions and fraction addition, while the number lines are recommended for their ability to help learners to understand that fractions are numbers, rather than one number over the other number and for helping learners to develop other fraction concepts (Van de Walle et al., 2013). Hence, the choice of using the two fraction models to teach fractions, for conceptual and procedural knowledge to Grade 8 learners during the intervention.

1.3 RATIONALE AND SIGNIFICANCE OF THE STUDY

Locke, Spirduso and Silverman (1993) argue that "every graduate student who is tempted to employ a qualitative design should confront one question, 'Why do I want to do a qualitative study?' and then answer it honestly" (p. 107). The choice of conducting this research study was strongly informed by my personal experience of teaching mathematics (and fractions in particular) in Namibian secondary schools over the last four years (since January 2011), immediately after I successfully completed my four year teachers' training degree - BEd Honours from the University of Namibia in December 2010. I noted that both teachers and learners found fractions difficult to teach and learn. My passion to improve the teaching of fractions in my classroom using models, began when I was introduced to the use of area models to represent fractions and fraction operations, in the mathematics education class (of 2014) in the final year of my BEd Honours programme with Rhodes University.

The rationale and focus of this research was three-fold. The first focus of this research was to generate data on how Grade 8 learners made sense of fractions as a concept and equally how they used procedures to make sense of fractions when comparing, ordering and adding fractions. Learners' understanding of fraction concepts were investigated by ascertaining how they made sense of area models and number lines, representing fractions. The sense making of fraction procedures was examined by determining how learners used appropriate procedures with understanding (shown by written explanations). The second focus of this research was to generate data to understand the processes by which the use of the area and number line models are useful in improving the teaching of fractions for conceptual understanding and procedural fluency.

A review of the literature revealed that this study (investigating the use of fraction models to teach fractions for conceptual understanding and procedural fluency) is the first of its kind in Namibia. Therefore, the findings of this research study will serve as a knowledge bank for mathematics teachers and future researchers in the field of mathematics education, seeking to improve the teaching of mathematics and fractions in our classrooms.

1.4 RESEARCH GOALS

The main research goal of this study is to investigate the use of area models and number lines to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions. In order to achieve this research goal, this research seeks to attain the following research objectives:

- To design a pre-test, post-test and learners' worksheets for the teaching intervention, according to the reviewed literature and published teaching resources for fractions.
- To describe the nature of Grade 8 learners' conceptual understanding of and procedural fluency with fractions, before the teaching intervention by administering and analysing the pre-test and interviews.
- To identify the changes in Grade 8 learners' conceptual understanding of and procedural fluency with fractions, after the teaching intervention by administering and analysing the post-test and interviews.
- To describe the influence of the teaching intervention on Grade 8 learners' conceptual understanding of and procedural fluency with fractions by observing and analysing the process of the teaching intervention.

1.5 RESEARCH QUESTIONS

In order to attain the research goals listed above, this research sought to answer the following questions:

- What was the nature of Grade 8 learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention?
- How did Grade 8 learners' conceptual understanding of and procedural fluency with fractions change during the teaching intervention?
- How did the teaching intervention influence Grade 8 learners' conceptual understanding of and procedural fluency with fractions?

1.6 OUTLINE OF THE DISSERTATION

Chapter Two presents the conceptual framework and literature that informs and shapes the analysis and interpretations of the findings of this study. Key concepts such as fractions, conceptual understanding and procedural fluency, fraction sub-constructs and fraction models used, are reviewed and defined.

Chapter Three describes the methodology and methods used to collect the data. It also describes the design of the teaching intervention, and the sampling techniques used for this research. Furthermore it discusses issues pertaining to data analysis, validity, ethical considerations and challenges encountered in the field during data collection.

Chapter Four presents data from five research instruments, namely: the pre-interview, pre-test, post-test, post-interview and recall interview. In addition, it details the analysis of this data in order to generate the research findings for the first two research questions (see Section 1.5).

Chapter Five presents the data from the two research instruments, namely: the learners' worksheets and transcripts of the lesson videos. Following this, it presents the analysis of this data to generate the research findings for the third research question (see Section 1.5).

Chapter Six presents a discussion of the research findings, linking the data analysis in Chapter Four and Five to the literature presented in Chapter Two. The research findings are presented as analytical statements related to each research question.

The final chapter, Chapter Seven, presents the summary of the research findings and recommendations of this research study.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter discusses the literature relating to the teaching and learning of fractions; in particular, teaching using representational models, such as area models and number lines. Firstly, it presents the scope of teaching of fractions in the Namibian curriculum. Secondly, it defines the concept of fractions, and describes the sub-constructs of fractions. Thirdly, the discussion focuses on the nature of difficulties associated with the learning of fractions, which are related to the two selected strands of mathematical proficiency, namely: conceptual understanding and procedural fluency. The chapter then presents and discusses the indicators of conceptual understanding and procedural fluency of fractions. Lastly, the chapter presents and discusses the literature on practically based approaches to teaching fractions for conceptual understanding; the role of multiple representations; and the role of the area models and number lines in teaching and learning fractions.

2.2 THE TEACHING OF FRACTIONS IN THE NAMIBIAN CURRICULUM

The field of mathematics education in the 21st century includes a significant quantity of research aimed at developing understanding of why learners still have difficulty with learning fractions and fraction operations (Pearn, 2007; Cooper, Wilkerson, Montgomery, Mitchell, Arterbury, & Moore, 2012; Siegler, Fazio, Bailey, & Zhou, 2013). Siegler et al. (2013) note that "learners have serious difficulties with learning … fractions and [even] many educated adults [including teachers and students lack] … adequate conceptual understanding of rational numbers [fractions]" (p. 1). A number of explanations for this have been given. One of these many explanations for the difficulty of learning fractions experienced by learners, is a result of the way in which fractions are taught at schools (Cooper et al., 2012). This identifies one contributing factor, the teaching of fraction concepts using rules, procedures and memorisation, instead of affording learners opportunities to develop their own understanding of fraction concepts (Cooper et al., 2012).

In Namibia, the timeframe of the formal teaching of fractions in schools is four years, beginning in Grade 5 and continuing up till Grade 8 (Namibia. Ministry of Education, 2010). Table 2.1 below, presents the learning content of fractions in these four grades.

Table 2.1: The learning content of fractions in the Namibian mathematics syllabi for Grade 5-8

Grade 5	
Topic:	**Specific learning objectives:**
Fraction vocabulary	1. Use the correct terminology of fractions such as numerator and denominator; 2. Treat denominators of fractions as divisors e.g. a/b as a ÷ b whereby b represents the whole number of parts into which a whole is to be divided.
Comparing and ordering	3. Compare and order fractions with the same denominators or numerators, e.g. 2/5 and 4/5 or 2/3 and 2/5; 4. Recognise equivalent fractions; 5. Compare and order fractions with different denominators including finding a common denominator, e.g. 1/4 , 1/3 and 1/5; 6. Locate fractions up to tenths as points on the number line.
Classification of Fractions	7. Identify proper fractions, improper fractions and mixed numbers; 8. Compare and order mixed numbers and improper factions in practical situations; 9. Convert between mixed numbers and improper fractions.
Fractional parts quantities	10. Calculate fractional parts of quantities.
Grade 6	
Equivalent fractions	11. Convert fractions to equivalents; 12. Write fractions in their simplest form.
Comparing and ordering	13. Compare and arrange fractions with the same and different denominators in a stated order by converting to a common denominator.
Addition and subtraction of fractions	14. Add and subtract fractions with the same and different denominators restricted to three terms /fractions; 15. Add and subtract mixed numbers.
Word problems	16. Solve two-step word problems involving addition (restricted to three terms) and subtraction with common fractions.
Grade 7	
The four basic operations and order of operations	17. Add and subtract common fractions including mixed numbers; 18. Multiply and divide fractions (including mixed numbers) by fractions; 19. Apply the correct order of operation BODMAS.
A quantity as a fraction of another quantity	20. Express one quantity as a fraction of another quantity.
Word problems	21. Solve three-step word problems involving common fractions.
Grade 8	
Common and decimal fractions	22. Convert common fractions to decimals by dividing the denominator into the numerator 23. Convert terminating decimals to common fractions in simplest form 24. Order and compare fractions 25. Multiply and divide quantities by common and fractions

(Namibia. Ministry of Education, 2010)

The analysis of the learning content of Grade 5-8 shows that the teaching of fractions is dominated by one representation – fraction symbols and one interpretation of fractions – the part-whole interpretation.

For instance, in Grade 5, fractions are introduced to the learners using the symbolic fraction notation, whereby learners have to identify the numerator and denominator of the fraction notation (see learning objective 1 in Table 2.1). The symbolic representation of fractions is also used for comparing and ordering fractions, classification of fractions and calculating fractional parts of quantities (see learning objectives 3-5 and 7-10, in Table 2.1). The part-whole interpretation of fractions appear to be related to two learning objectives, namely: number 2 and 6, which are to regard the denominator of fractions as the number of parts into which a whole is divided and to locate fractions up to tenths as points on the number line. For instance, the analysis of the textbook for Grade 5 (Maths for Life) shows dominance of the part-whole interpretation of fractions, which comprises the use of the partly shaded circles and rectangles including the fraction chart, to identify fractions whose numerator is smaller than the denominator only; and the use of pre-partitioned number lines for identifying and locating only those fractions, whose numerator is less than the denominator.

In Grade 6, the analysis of the textbook in use (Hands-on Maths) shows dominance of the part-whole interpretation of fractions, which includes the use of the fraction chart to identify equivalent fractions (see learning objective 11) and introduces the comparing and ordering of fractions with the same denominators only (see learning objective 13). However, the teaching of fractions in this grade emphasises the use of procedures (such as lowest common denominator method) and symbolic notation of fractions to teach equivalent fractions, comparing and ordering of fractions, addition and subtraction of fractions, as well as to solve word problems (see learning objectives 12-16).

The use of symbolic representation of fractions continues to dominate the teaching of fractions in Grade 7 and 8. For instance, the analysis of textbooks of the above mentioned grades shows the use of procedures for teaching the learning objectives 17-25 in Table 2.1.

2.3 FRACTIONS

2.3.1 Definitions of fractions

The term “fractions” can be used to refer to “any way of representing rational numbers, such as percentages, decimals and common fractions” (Hansen, 2015, p. 9). The present study is concerned with the development of common fractions. Common fractions refer to a representation of rational numbers in the form a/b (Hansen, 2015). In this study, the terms fractions and common fractions are used interchangeably. Common fractions can be classified into three groups, namely: proper fractions, improper fractions and mixed numbers. In the Namibian school mathematics textbooks, the three types of common fractions are defined as follows:

- Proper fraction is a fraction with a numerator smaller than the denominator e.g. 2/3;
- Improper fraction is a fraction with a numerator bigger than the denominator e.g. 3/2;
- Mixed number is a whole number together with a fraction e.g. 1 ½ (Hambata, Roos, & Van der Westhuizen, 2015, p. 49).

The analysis of the Namibian school textbooks of mathematics, show that the teaching of common fractions gives emphasis to converting improper fractions into mixed numbers and vice versa, rather than showing the conceptual relationship between improper fractions and mixed numbers. The textbooks use symbolic notation for identifying proper fractions and improper fractions, but they do not give conceptual interpretations, to explain why proper fractions have numerators less than denominators, and why improper fractions have numerators greater than denominators. The present study seeks to establish conceptual interpretations of why proper fractions have numerators less than denominators and improper fractions have numerators greater than denominators, through the teaching intervention using the two representations of fractions, namely: the area model and number lines.

2.3.2 Interpretations of fractions

For learners to make sense of symbolic representations of common fractions, they need to understand that the interpretation of symbolic fraction notation depends on the context in which the fraction is used (Cooper et al., 2012; Hansen, 2015). According to Cooper et al. (2012), Clarke, Roche and Mitchell (2011), McNulty, Editor and Morge (2011) and Hansen (2015), a fraction can be interpreted as a part of a whole or set, a measure, a ratio, a division, and an operator. This grouping of fractions as sub-constructs was developed by Behr, Lesh, Post and Silver (1983) and Kieren (1988). The interpretations of fraction sub-constructs are described below as follows:

- Part-whole interpretation of fractions involves partitioning of continuous quantity or a set of discrete objects into a number of equal parts. In this interpretation, the numerator must be smaller than the denominator. This interpretation is the most common interpretation used in elementary school exercise books and is the one children use consistently more often than other interpretations;
- Ratio interpretation of fractions involves writing two quantities as a ratio comparison rather than a number;
- Operator interpretation of fractions is when a fraction is applied as a function to a number, set or objects; e.g. showing 3/4 of a pie chart or finding 3/4 of 24;
- Quotient interpretation of fractions is the result of a division; e.g. 3÷4=3/4;
- Measure interpretation of fractions involves using the given unit interval to measure any distance from the origin; locate a number on a number line; and identify a number represented by a point on the number line (Hansen, 2015).

The present study is concerned with the development of part-whole interpretation of fractions and the measure interpretation of fractions.

Hannula (2003) describes the part-whole sub-construct as being the fundamental key of rational numbers, because it is used so much in teaching and because of its strong use to interpret fraction symbols. This is supported especially in the U.S, New Zealand and Australia, from which this literature on fraction learning was drawn. It appears that the literature on teaching and developing of fraction concepts, is well documented and more researched in the three countries mentioned earlier, than the rest of the world, including Africa and Namibia in particular. However, on its own, the part-whole interpretation of fractions is regarded as an insufficient foundation to develop a good conceptual understanding of fractions (Clarke et al., 2011).

Hansen (2015) indicates that the measure interpretation of fractions is of "primary importance for developing an understanding of fractions" (p. 11). To support her claim, she made reference to a study done by Jordan, Hansen, Fuchs, Siegler, Micklos and Gersten (2013) which showed that a fraction intervention study which emphasised the number line or measurement interpretation of fractions increased both the fraction concepts and procedures of fourth-graders, more than classroom instruction that only focused on part-whole interpretation of fractions.

According to Hannula (2003), a survey of 3067 Finish 5th and 7th graders, assessed students' understanding of fractions, by asking the students to find ¾ of an eight piece bar and its location on a number line. The results of this survey showed that the part-whole interpretation of fractions dominated students' thinking and that students had difficulties in perceiving a fraction ¾ as a number on the number line. Further discussion of the students' thinking of the part-whole and measure interpretations of fractions, will occur later in this literature review.

2.3.3 Importance of learning fractions

The inability of learners to learn fractions is a serious, global educational concern (Pearn, 2007; Siegler et al., 2013). According to Siegler et al. (2013), a recent National Assessment of Educational Progress (NAEP) test was written by a large nationally representative sample of Grade 8 learners in the USA, but only 50% of those learners could correctly order the fractions 2/7, 1/12, 5/9 in ascending order. This same problem of a distorted understanding of fraction magnitudes, was observed in countries such as China, Taiwan and Japan, which are internationally ranked as being the best nations in terms of the quality of mathematics learning and teaching (Siegler & Lortie-Forgues, 2015). In addition, the teachers of mathematics in the USA, China, Japan and Taiwan, identify the lack of fraction understanding as one of the two largest problems hindering their learners' algebra learning (Siegler et al., 2013; Siegler & Lortie-Forgues, 2015). Thus, the USA National Mathematics Advisory Panel (BMAP) of 2008, selected fraction proficiency as one of the most important foundational skills that should be taught in schools (Siegler, et al., 2013; Siegler & Lortie-Forgues, 2015) for their citizens to become numerate and as a pre-condition for attaining an increase in student algebraic achievement (Siegler, et al., 2013).

Fraction knowledge is also one of the areas of mathematics that seems important for later success. This was evident in a study done on Grade 5 learners in both the UK and USA, where fraction knowledge in that grade predicted learners' algebra knowledge and overall mathematics achievement in the tenth grade (Siegler & Lortie-Forgues, 2015). Additionally, fraction knowledge is claimed to be "essential for a wide range of occupations beyond science, technology, engineering and mathematics fields, including nursing, pharmacy, automotive technician, stone mason, and tool and die maker" (Siegler & Lortie-Forgues, 2015, p. 1). Unfortunately, "the fraction knowledge of many children, adolescents and adults is still poor" (Siegler & Lortie-Forgues, 2015, p. 1).

2.3.4 Misconceptions associated with learning fractions

A number of authors (Stafylidou & Vosniadou, 2004; Siegler et al., 2013; Siegler & Lortie-Forgues, 2014, 2015) have identified common misconceptions that learners display when learning fractions. These often relate to erroneous assumptions that properties of whole numbers, are the properties of all numbers, including fractions. An example given by Siegler et al. (2013) is that whole numbers have unique successors, are represented by a unique symbolic representation and are countable in their natural order (Siegler et al., 2013). Yet, "fractions are indefinitely divisible with infinitely many rational numbers between any two given fractions" (Siegler et al., 2013, p. 14). Unfortunately, many learners do not easily grasp the concept of indefinite divisibility (Siegler et al., 2013). Therefore, learning fractions requires learners to conceptually differentiate properties of natural numbers, from properties of rational numbers (Siegler & Lortie-Forgues, 2015). Siegler and Lortie-Forgues (2014, 2015) also view magnitude as the fundamental property of real numbers, stating that every real number has a magnitude with a unique position on a number line and that it is important for the learners and teachers to respect this, as the only common property of all numbers.

A second misconception associated with fractions that was investigated by Siegler et al. (2013) is when learners confuse fraction arithmetic procedures. For instance, for fraction addition and subtraction with the same denominator, that denominator is maintained in the answer, but this does not hold for either fraction multiplication or division. Another problem that many learners face with fraction arithmetic, is that they do not really understand when and why common denominators are maintained. According to Siegler et al. (2013), the choice of fraction arithmetic strategies is strongly linked to learners' constrained fraction conceptual knowledge. This constraint could be minimised by improving learners' understanding of fraction magnitudes (Siegler et al., 2013; Siegler & Lortie-Forgues, 2014).

Stafylidou and Vosniadou (2004) identify a misconception displayed by some students relating to comparing and ordering fractions that involved students ordering fractions either on the basis of the size of the numerator only, or of the denominator only. Their research study tested 200 students, ranging in age from 10 to 16 years, on comparing fractions and their research results established that students ordered fractions either on the basis of the size of the numerator only or of the denominator only. They suggest that students who ordered fractions on the basis of the numerator, appeared to ignore the denominators and ordered fractions in such a way that as the numerator of a fraction increased, the fraction itself also increased, while students that ordered fractions on the basis of the denominator, appeared to ignore the numerators and ordered fractions in such a way that as the denominator increased, the fraction itself also increased. Stafylidou and Vosniadou (2004) indicate that students commit these errors by simply transferring whole-number ideas to fractions (i.e. larger numbers mean greater magnitude) and working with individual fraction components (numerator, denominator) rather than a fraction as a single entity. Siegler et al. (2013) indicate that this type of confusion arises from instruction emphasising a single interpretation of fractions; that is, viewing fractions exclusively in terms of part-whole relations.

2.4 TWO STRANDS OF MATHEMATICAL PROFICIENCY

This research study focuses on two selected strands of mathematical proficiency. Kilpatrick et al. (2001) describe mathematical proficiency as "a composite, comprehensive view of successful mathematics learning" (p. 5). They indicate that the concept of 'mathematical proficiency' consists of five strands which are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These strands were defined as follows:

- *Conceptual understanding* – comprehension of mathematical concepts, operations, and relations;
- *Procedural fluency* – skills in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification;
- *Strategic competence* – ability to formulate, represent and solve mathematical problems;

- *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Kilpatrick et al., 2001, p. 5).

Kilpatrick et al. (2001) stress that the five strands are "interwoven and interdependent" (p. 5). Still, the present study focuses *only* on two strands of mathematical proficiency, namely: conceptual understanding and procedural fluency. The selected strands serve as the conceptual framework of this research study. Conceptual understanding is fundamental to the study, because it relates to the learners' capacity to "see connections among concepts and procedures and to explain why some facts are consequences of others" (Kilpatrick et al., 2001, p. 119). In addition, the research was concerned with learners' calculations and learners' capacity to see connections and differences between methods of calculating. Kilpatrick et al. (2001) state that learning without understanding makes learning of new topics hard, since there is no network of previously learned concepts and skills to link to a new topic. This is referred to as 'compartmentalization of procedures' where learners could believe that even slightly different problems require different procedures (Kilpatrick et al., 2001). Another effect of compartmentalisation of procedures, is when the learners tend to "have one set of procedures for solving problems outside of school and another they learned and use in school – without seeing the relation between the two" (Kilpatrick et al., 2001, p. 123). For these reasons, conceptual understanding and procedural fluency were selected as the conceptual framework of this study.

A number of authors (Balka & Harbin, n.d; Gabriel et al., 2012; Wiggins (2014) defines conceptual understanding (mathematical understanding) differently. Wiggins (2014) defines conceptual understanding as "the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from" (p. 1). Balka and Harbin (n.d.) suggest that students demonstrate conceptual understanding in mathematics in this way:

> when they provide evidence that they can recognise, label, and generate examples of concepts; use and interrelate models, diagrams, manipulatives, and varied representations of concepts; identify and apply principles; know and apply facts and definitions; compare, contrast, and integrate related concepts and principles; recognise, interpret, and apply the signs, symbols, and terms used to represent concepts. (p. 2)

Gabriel et al. (2012) define conceptual knowledge (or conceptual understanding) as "the explicit or implicit understanding of the principles ruling a domain and the interrelations between the different parts of knowledge in a domain" (p. 137). This includes "knowledge of central concepts and principles and their interrelations in a particular domain" (Gabriel et al., 2012, p. 137). According to Balka and Harbin (n.d.), conceptual understanding is a reflection of students' abilities to reason in settings that demand the careful application of concept definitions, relations, or representations of either. Conceptual understanding in mathematics makes "learning [of] skills easier, less susceptible to common errors, and less prone to forgetting" (Kilpatrick et al., 2001, p. 123). Kilpatrick et al. (2001) explain that it also enables students to:

> know more than isolated facts and methods; understand why a mathematical idea is important and kinds of contexts in which it is useful; have organized their knowledge into a coherent whole which enable them to learn new ideas by connecting those ideas to what they know; remember with ease, reconstruct and use facts and methods learned; monitor what they remember and try to figure out whether it makes sense; attempt to explain the method to themselves and correct it if necessary; verbalize connections among concepts and representations; represent mathematical situations in different ways and knowing how different representations can be useful for different purposes; see connections among concepts and procedures and explain why some facts are consequences of others; and avoid many critical errors in solving problems, particularly errors of magnitude. (pp. 118-119)

According to Kilpatrick et al. (2001), procedural fluency refers to "knowledge of procedures, knowledge of when and how to use them appropriately, and skills in performing them flexibly, accurately, and efficiently" (p. 121), while Gabriel et al. (2012) see procedural knowledge (or procedural fluency) as the "knowledge of symbolic representations, algorithms and rules" (p. 138). In the domain of numbers, procedural fluency is "especially needed to support conceptual understanding of place value and the meanings of rational numbers" (*ibid.*). In mathematics, procedural fluency "supports the analysis of similarities and differences between methods of calculating" (*ibid.*). These methods include "mental methods for finding certain sum, differences, products, or quotients, as well as methods that use calculators, computers or manipulative materials such as blocks, counters or beads" (*ibid.*) in addition to written procedures.

According to Kilpatrick et al. (2001), students need procedural fluency to enable them to:

> be efficient and accurate in performing basic computations both mentally, and with pencil and paper; estimate the results of a procedure; illustrate the link between conceptual understanding and procedural fluency; see that procedures can be developed that will solve entire classes of problems, not just individual problems; and acquire a certain level of skills required to learn many mathematical concepts with understanding. (pp. 121-122)

Without conceptual understanding, students would learn procedures without understanding, which would make it difficult for the students to be engaged in activities which are meant to help them to understand the reasons underlying the procedures (Kilpatrick et al., 2001). For instance, based on the results of an experimental study of 5th grade students, "students who first received instruction on procedures for calculating the area and perimeter followed by instruction on understanding those procedures did not perform as well as students who received instruction focused only on understanding" (Kilpatrick et al., 2001, p. 122).

2.5 DEVELOPING CONCEPTUAL KNOWLEDGE AND PROCEDURAL KNOWLEDGE OF FRACTIONS

In the context of fractions, conceptual knowledge can be defined as "a combination of the general properties of rational numbers (such as the principles of equivalent fractions), the understanding of the roles of the numerator and the denominator, and the understanding of the global fraction magnitudes" (Gabriel et al., 2012, p. 138). Also, procedural knowledge of fractions is about "knowing how to calculate the lowest common denominator to add or subtract fractions with different denominators" (Gabriel et al., 2012, p. 138).

Conceptual knowledge of fractions includes "understanding that fractions represent parts of an object or a set of objects, that they are represented by fraction symbols [a/b], and that fractions are [indeed] numbers that reflect magnitudes ... [and] can be ranked from smallest to largest" (Jordan et al., 2013, p. 5). Conceptual understanding of fractions also includes understanding the meaning of the fraction numerator and denominator, as well as their relationship to each other in a holistic manner, for both measure and part-whole constructs (Wiest, Thomas, & Amankonah, 2014). Apart from being able to use and make visual representations (fraction bars and number lines) of fractions less than 1, equal to 1 or greater than 1, learners showing developing conceptual understanding of fractions are expected to be competent with partitioning of circles, fraction bars and number lines, with representing the given fraction notation, and developing more than one strategy of comparing, ordering or adding fractions, including the use of benchmarking (Wiest et al., 2014).

Procedural fluency of fractions includes any form of computation with fractions (Jordan et al., 2013; Hansen, 2015), such as calculating the common denominator (Gabriel et al., 2012), comparing and/or adding fractions, or the use of rules to compare fractions with the same numerators, fractions with the same denominators or fractions with different numerators or denominators. Unfortunately, many learners seem to apply procedures such as the lowest common denominator of fractions, without fully understanding the underlying concepts (Gabriel et al., 2012). Gabriel et al. (2012) state that instruction that includes both concepts (like fractions) and procedures, leads to improved procedural knowledge, proven by the fact that learners that were found to possess knowledge in both, outperformed others with only one of the forms of knowledge (Pantziara & Philippou, 2012, p. 67). They further argue that evidence from mathematical studies has shown that development of conceptual knowledge should precede procedural knowledge.

According to the findings of a study that looked at the relations among fraction knowledge of fifth-graders in the U.S. (Hansen, 2015), knowledge of fraction concepts was found to have strongly influenced the outcomes of learners' fraction procedures. A strong understanding of fraction concepts allowed the children to understand and use appropriate procedures to solve fraction problems by avoiding less flawed arithmetic and also helping them to deal with forgetting the procedures (Hansen, 2015).

While no literature on developing learners' conceptual understanding and procedural fluency of fractions in Namibia was found, a similar intervention study to this present study was carried out in Belgium with Grade 4 and 5 learners (Gabriel et al., 2012). The intervention study in Belgium induced learners to manipulate, compare, and evaluate fractions through playing games in the classroom and these games helped them to successfully understand numerical magnitudes of fractions, i.e. by linking magnitudes to their associated fraction notation (Gabriel et al., 2012). However, the Belgium's study intervention failed to use the developed conceptual understanding of fractions of learners, to develop procedural knowledge of fractions. Hence, this research study investigates the use of fraction models to develop conceptual understanding and procedural fluency of fractions of Grade 8 learners in Namibia.

2.6 TEACHING APPROACHES FOR CONCEPTUAL UNDERSTANDING

Way (2011) and Canterbury (2007) propose a variety of non-traditional fraction instructional approaches, to develop conceptual understanding of fractions by de-emphasising rote learned rules and procedures of solving fraction problems. In order to develop fraction sense making in learners, Way (2011) and Clarke, Roche and Mitchell (2011) urge teachers to be aware that fraction concepts develop over time and therefore teachers need to be patient and passionate (Mack, 1990), and give their learners sufficient exploration and experimentation to learn fraction concepts, rather than teaching the recipe of implicit fraction algorithms. The following teaching approaches for developing conceptual understanding of fractions served as a guideline for designing the teaching intervention of this research.

1. Teachers are advised to teach fractions by giving "emphasis to the meaning of fractions than on procedures for manipulating them" (Clarke et al., 2011, p. 29; Clarke & Roche, 2011, p. 4). This approach challenges the norm of many curriculum documents for teaching fraction arithmetic, by teaching learners to understand the conceptual meanings of fractions and reason proportionally (Clarke et al., 2011). In this research study, every worksheet of the intervention focuses on developing the meanings of fractions.

2. Using area representations demonstrates to learners that common fractions are representations of equal parts of a whole. Learners are also expected to name fractions both verbally and symbolically (Way, 2011, p. 155). According to Clarke et al. (2011) teachers can attain this by developing "a general rule for explaining the numerator and denominator of a fraction" (p. 29). Clarke et al. (2011) encourages teachers to avoid defining the denominator as just the number of parts a whole is divided into and the numerator as the number of parts to take, count or shade, as they believe that such definitions may lead to fraction misconceptions, as they are only true for proper fractions (fractions between 0 and 1) but not for improper fractions (fractions greater than 1).

3. This research study emphasises the use of appropriate language when labelling fractions, to allow learners to recognise fractions as numbers (magnitudes) and to distinguish between the digits that refer to the number of parts or the size of the parts (Way, 2011). Clarke et al. (2011) indicate that to read three-quarters as "three-fours",

"four-threes", "three over four", "three divided by four" or "three out of four" is very confusing and could encourage learners to think of fractions as two separate unrelated whole numbers or they could fail to tell which digits represent the number of parts or the size of the parts (Clarke et al., 2011).The present study discourages learners from reading fractions as two unrelated whole numbers by encouraging the use of appropriate names (e.g. three quarters) to read fractions.

4. Emphasis can be made of fractions as magnitudes by making extensive use of number lines in representing fractions. Clarke et al. (2011) explain that the use of number lines has many advantages. Firstly, number lines help learners to see how whole numbers and fractions are related. Secondly, they provide a way to see how 5/3 is the same as 1 2/3 and 6/2 is the same as 3 (Clarke et al., 2011). Thirdly, number lines make it easier for the learners "to understand the density of rational numbers" (Clarke et al., 2011, p. 30). Pantziara and Philippou (2012) also suggest the extensive use of number lines for fraction investigations to help learners to understand the relative size of fractions and to think of a fraction as a single number.

5. Focusing the teaching of fractions on making "sense of the size of fractions [and fraction types] in relation to one whole" (Way, 2011, p. 155) develops this essential skill in fraction understanding (Cramer et al., 2008). In order to achieve this goal, Way (2011) advises teachers to use strategies such as benchmarking, use of equal number lines or fraction bars, to order fractions from smallest to biggest. Teachers are also advised to teach fractions by using examples of fractions less than 1, equal to 1 and greater than 1. This is deemed necessary for the learners to understand "fractions as values that come between whole numbers (or equivalent to whole numbers)" (Van de Walle et al., 2013, p. 357). Moreover, Van de Walle et al. (2013) explain that teachers are advised to help the learners to make sense of fraction symbols by posing questions such as:

> What does the numerator in a fraction tell us? What does the denominator in a fraction tell us? What might a fraction equal to one look like? How do you know if a fraction is greater than or less than 1, … [is] greater than or less than 2? (p. 356)

6. Clarke et al. (2011) suggest that it is very important for teachers to "take opportunities early to focus on improper fractions and equivalences" (p. 30). The term improper fraction is used to "describe fractions that are greater than one" (Van de Walle et al.,

2013, p. 357). Clarke et al. (2011) suggest that developing the meaning of improper fractions and equivalences can be encouraged by the effective use of number lines and area representations.

7. Using a variety of models to represent fractions, helps learners to "visualise, estimate and create representations of fractions by partitioning wholes" (Way, 2011, p. 156). According to Way (2011), teachers need to keep to a minimum the use of stereotyped pre-partitioned representations of fractions, to avoid automatic responses. Instead they should allow learners to think creatively and visualise the parts before partitioning the representations by themselves. For instance, when learners are to compare two fractional parts, they need to appreciate the fact that the units need to be of the same size (Clarke et al., 2011). Put differently, for the learners to make sense of symbols, requires making connections to the visual representations. For instance, the use of two different models to represent the same improper fraction, is said to be helpful for learners to notice a pattern "that actually explains the algorithm for moving between mixed fractions and fractions greater than 1" (Van de Walle et al., 2013, p. 357). It further states that rushing learners into a standard algorithm of converting mixed fractions into improper fractions, can seriously interfere with making sense of the relationship between the improper fractions and their equivalence (Van de Walle et al., 2013).

8. Teachers are encouraged to teach in a way that their learners can "link fractions to key benchmarks" (Clarke et al., 2011, p. 31). According to Clarke et al. (2011), benchmarking is one of the creative strategies that successful learners often use when working on tasks of comparing the relative size of fractions. Benchmarking refers to the capacity to relate the size of fractions of interest, to known magnitudes such as 0, ½, and/or 1 (Clarke et al., 2011). Furthermore, Clarke et al. (2011) state that allowing successful students to share their strategies with their classmates, can help other learners to solve problems of comparing the relative size of fractions and ordering fractions using benchmarking.

9. Teachers are encouraged to take the opportunity to interview students one-to-one during regular classroom activities in order to gain awareness of students' thinking and strategies. Clarke et al. (2011) and Siegler et al. (2010) state that the use of one-to-one interviews is very useful for gaining insights into the learners' thinking and the

way learners attempt to make sense of fractions, as well as to confirm their understanding. Furthermore, Clarke et al. (2011) suggest that teachers should always use examples and activities which can increase learners' engagement and thinking about fractions, in a particular way. Way (2011) states that these approaches, if well implemented, are feasible to develop learners' mental images of fractions and apply their conceptual understanding to complex concepts and processes of fractions.

2.7 USING MODELS FOR TEACHING FRACTIONS

This sub-section presents a general discussion of the types of models used for teaching fractions for conceptual understanding. It also discusses the advantages and principles for effective use of multiple representations to teach fractions. Lastly, it presents and discusses the advantages, challenges and principles for effective use of number lines and area models to teach fractions.

"There is a range of models commonly used to support fraction instruction" (Harvey, 2011, p. 334) such as sets of discrete objects, number lines (linear model), double number lines and area models, such as circles and rectangles (Watanabe, 2002; Harvey, 2011). The typical common fraction models used in elementary and middle school mathematics textbooks are the linear model, the area model and the discrete model (Watanabe, 2002). According to Harvey (2011), teachers need to consider the effectiveness of models as a first priority, whenever they are to choose the model for fraction instruction. Important attributes of effective models include, but are not limited to, the relative length for linear models, relative area for two-dimensional models and relative number in the set model (Harvey, 2011).

In this study, the terms representations, physical or visual representations and models are used interchangeably, as different authors in the literature have used them. For instance, Watanabe (2002) indicates that the two words model and representations, are not synonymous, and chooses to use them interchangeably. According to Watanabe (2002), the "reason for this ambiguity is that both representation and model have several different meanings and share some meanings" (p. 457). He describes a model as anything that can be used to represent a mathematical idea such as "a scale model of an object, a series of equations that mathematically model a physical phenomenon, a demonstration, something that illustrates or exemplifies a mathematical concept, concrete materials used in instruction, and so on" (p. 457). In the context of this study, the term (fraction) model refers to the instructional materials used to teach fraction concepts namely, number lines and area models, which include circles and rectangles (fraction bars). According to The National Council of Teachers of Mathematics (2000) the term representation refers:

> both to process and to product – in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. ... Moreover, the term applies to processes and products that are observable externally as well as to those that occur "internally" in the minds of the people doing mathematics. (p. 67)

The use of visual models in mathematics, especially number lines and area models to represent fractions, has been recommended by many mathematics organisations, including the National Mathematics Advisory Panel's Critical Foundation for Algebra (Council of Chief State School Officers, 2010), Texas Higher Education Coordinating Board (2009) and NCTM's Principles and Standards for School Mathematics (2000). All these mathematical organisations have made calls for schools to train learners to fluently and flexibly use multiple representations of mathematical ideas (Gray, 2014). According to Gray (2014), the use of fraction models in the middle grades is considered a key to learners' success to master or "conceptually anchor the algorithms used to work with fractions" (p. 7) and for learners to make connections among models which eventually deepen their conceptual understanding (Gray, 2014). Cramer et al. (2008) concur with Gray (2014), stating that representations play a vital role in learning fractions as they allow students to understand mathematical concepts and relationships as well as to make sound mathematical arguments to convince one's self and others. According to Canterbury (2007), learners use fraction representations in four different ways to solve fraction tasks. Firstly, to communicate and organise their mathematical thinking and reasoning. Secondly, to obtain a visual representation of the task. Thirdly, to check the accuracy of their work, and lastly, to have a clear picture of their task.

Rau, Aleven and Rummel (2013) underscore that fractions are one of the mathematics domains in which multiple graphical representations (e.g. circles, rectangles and number lines) are extensively used, because they play key complemenatry roles to enhance conceptual aspects of fractions. Interestingly, they stressed the point that, like any other learning tools, multiple graphical representations (MGRs) do not enhance learning by themselves. They point out that learners can only benefit from the use of MGRs by having a good understanding of each representation, being fluent in using those representations and making connections between them (Rau et al., 2013). For instance, in one of their experimental studies, results indicated that learners who worked with MGRs of fractions, outperformed learners who worked with a single graphical representation (SGR). They indicated that this difference was made when learners were asked to explain how the graphical representations (e.g. quarter of a circle) relate to the symbolic representation (e.g. ¼). In addition, their results also indicated that affording learners opportunities to relate the numerator and the denominator to each graphical representation, helped learners to benefit a lot from the use of MGRs. In order to ensure success on the use of MGRs, Rau et al. (2013) carried out classroom experiments with over 3000 learners in the fourth till the sixth grades. Based on this research, they outline the following principles on the effective use of MGRs for fraction learning below.

The first principle states: "use MGRs to support conceptual learning". As described earlier, results of teaching experiments indicate that the use of MGRs lead to a better learning of robust conceptual understanding, than SGR. This depends on the type of instructional support and connections between MGRs that learners can achieve. In addition, they indicate that the pairing of text or symbols with a SGR also leads to better learning (Rau et al., 2013). The second principle states that teachers should use "use prompts to support understanding of graphical representations". This strategy again is deemed necessary to help learners to conceptually relate the concepts of the denominator and numerator to each representation. In a classroom experiment with 132 learners, their results (Rau et al., 2013) showed that learners benefited from MGRs after they were prompted. The third principle calls for interleaving topics to enhance understanding of graphical representations. In the same way, results of a classroom experiment with 158 students suggested that alternating between topics while blocking representations, is a better way to enhance learners' understanding of MGRs. The fourth principle is about interleaving representations to support fluency with graphical representations and the last principle is about helping learners to make connections between MGRs. Findings in their classroom experiments with 599 learners strongly suggest that "fluency-building support helps learners to gain experince in relating MGRs based on their perceptual properties" (*ibid.*).

Based on the literature (Van de Walle et al., 2013), the effective use of fraction models can help learners to "clarify ideas that are often confused in a purely symbolic model" (p. 342). The same paper also stipulates that at times the use of two different and appropriate models coupled with asking learners to make connections between the models, is an effective way of broadening and deepening both teachers' and learners' understanding of fractions. This is recommendable, because different models are known to offer different learning opportunities. For instance, an area model is believed to be better at helping learners to visualise parts of the whole, while a linear model is good at proving that between any two fractions there is always another fraction. Furthermore, some learners are said to be better at making sense of one representation than others (Van de Walle et al., 2013). Therefore, it is recommended that teachers give their learners ample experience with fractions, using real world-contexts that are meaningful to the learners. This is belived to be more helpful to learners especially when one context can align well with one representation, than others. For example, a linear model is considered better than the area model, if learners are to determine who walked the farthest (Van de Walle et al., 2013).

2.7.1 Using area models for teaching fractions

In this study, two area models (circles and rectangular bars commonly referred to as fraction bars) were used for fraction instruction. Both models are described as being good at emphasising "the part-whole concept and the meaning of the relative size of a part to the whole" (Van de Walle et al., 2013, p. 343). For instance, Cramer et al. (2008) found that the fraction circle model is the most powerful part-whole model for building mental images of the relative size of fractions and fraction addition. They added that the use of fraction circles is effective at enhancing learners' ability to see that the more the number of equal pieces the circle is divided into, the smaller the pieces. This is literally described as the inverse relationship between the denominator and the size of the fractional pieces (Cramer et al., 2008).

The pictorial (visual) representations of part-whole area models, are described as the simplest representation because the number of total equal parts in the area model, matches and is associated with, the fraction denominator, while the shaded parts are associated with the numerator (Wong & Evans, 2011). Van de Walle et al. (2013) state that "a fraction is based on parts of an area" (p. 343). Van de Walle et al. (2013) suggest that it is always good and also easy, to introduce fraction instruction using the area model, since it is well connected with equal sharing and partitioning. Wong and Evans (2011) emphasise the importance of learners ensuring that the whole is divided into equal parts, to make it easier to name fractions appropriately. This can be achieved by ensuring that learners practice partitioning with area models. Partitioning is defined as a process of dividing a shape into equal sized parts (Van de Walle et al., 2013). Van de Walle et al. (2013) stipulate that whenever learners are to partition area models, teachers should re-inforce that (a) fractional parts must be made of the same size and (b) the number of equal-sized parts within the unit or whole, determines the fractional amount e.g. the fourths.

However, the research findings of the study by Kerslake (1986) with 12 to 14 –year old students, suggests that the use of part-whole models, if used as the only interpretation of fractions, does not prompt learners to think of fractions as numbers, but only as a shaded number of parts over the total number of pieces in a shape or quantity. Amato (2005) stipulates that the counting process to name a fraction, does not prompt learners to think of fractions as parts of the whole, but rather to think of the fraction as a pair of two whole numbers. A direct impact of this type of learning, is that learners have shown difficulties in identifying a proper fraction on a number line showing more than one unit of length (Amato, 2005). Another difficulty shown by the learners is identifying the "unit in part-whole diagrams showing more than one unit" (Amato, 2005, p. 49) or fractions greater than one (e.g. reading 7/10 instead of 7/5). The use of separate area models to show the addition of two proper fractions or show the sum that is greater than one unit, is also described as to be cumbersome to those learners (Amato, 2005). Pirie and Kieren (1994) appear to agree with Amato (2005), when a 10 –year old child in their study achieved a new understanding of the process of adding halves and thirds (with a total less than a unit) by representing them on the area models before re-partitioning both area models into sixths. Similarly, Cramer et al. (2008) claims that:

> Fraction cirles vividly demonstrate the need for finding common denominators when adding and subtracting fractions and that fraction circles show the steps to exchanging given fractions with equivalent ones with common denominators. (p. 496)

In their work, Cramer et al. (2008) could not point out how the fraction circles might have helped learners to add fractions with a sum greater than 1. Thus, the findings of Amato (2005) on learners having difficulties in using separate area models to show the sum greater than one unit, is still unchallenged. To conclude, Cramer et al. (2008) indicate that the majority of learners would require extended periods of time and much practice if they are to fully grasp the idea of using circles to show the process of adding fractions. On the other hand, Amato (2005) conducted research to investigate the effects of understanding fractions as numbers, by using multiple representations to teach mixed numbers. Results of her study strongly suggest that the use of multiple representations to show fractions equal to 1 and mixed numbers, do help learners to understand fractions as numbers and realise that proper fractions of the mixed number notations are numbers less than 1.

2.7.2 Using the number line model for teaching fractions

The use of number lines, which is based on the measure construct of fractions for instruction, is considered very effective for helping learners to "co-ordinate information provided pictorially by the marked line together with the numbers which give information about scale" (Harvey, 2011, p. 335). Number lines are recommended for their ability to help learners to understand that fractions are numbers rather than one number over the other number and for helping learners to develop other fraction concepts (Van de Walle et al., 2013). Unlike area models, linear models are said to be not widely used in elementary and junior high schools, yet they are the most challenging and essential tools that should be emphasised more in the teaching of fractions, for developing learners' conceptual understanding (Larson, 1987; Van de Walle et al., 2013). "Like with whole numbers, the number line is used to compare the relative size of numbers" (Van de Walle et al., 2013, p. 345).

Mitchell and Horne (2011) indicate that the conventions for reading and drawing number lines to represent fractions, as described above by Harvey (2011), are sometimes perplexing to the learners because they may use one or two of their part-whole interpretations to solve fraction tasks involving number lines as discussed below.

The first part-whole interpretation of the measure construct is the "part-whole segment of a line", often called the "measure sub-construct part-whole" (Mitchell & Horne, 2011, p. 53) in which a number line is interpreted as a segment of a rectangular object such as a paper strip. The second interpretation is to think of the length of a number line as a simple line that shows the whole and a fraction like 1/3 would be thought of as a third of the way along a line, where the left hand edge of the line is assumed to be a 0 point (Mitchell & Horne, 2011). However, learners need to think of a point on the number line representing 1/3, as a number located between 0 and 1 and 7/4 being located between 1 and 2 on the number line with a pre-set 0. In addition, Pantziara and Philippou (2012) note that the ability to identify fractions as points on the number line, can allow learners to think of fractions as single numbers.

2.7.3 Misconceptions associated with the number line tasks

Mitchell and Horne (2011) identify three misconceptions in the use of learners' strategies for solving number line tasks. These misconceptions include: limited part-whole understanding; assuming a decimal number line; and counting the 0 point. According to findings of a research study by Mitchell and Horne (2011), the most rampant misconception among learners is limited part-whole understanding, whereby learners locate fractions less than 1, after 1 on the number line, or locate the fraction 1/n at (1/n)th of the distance between 0 and 2 (Amato, 2005). The second misconception was using decimals such as tenths to identify fractions on the number line, while the third misconception was counting the lines including the 0 point instead of the spaces, which make learners obtain fraction components (numerators and denominators) having one more part, for instance, obtaining 5/6 instead of 4/5.

Van de Walle et al. (2013) make it clear that locating a fractional value on a number line is really challenging, yet a skill that every learner must be able to do. The same paper outlined four common errors learners are likely to make. These include using incorrect fraction notation, changing the unit, counting the tick marks rather than the space between tick marks, and counting the tick marks that appear without noticing any missing (Van de Walle et al., 2013). Larson (1987) appears to agree with Van de Walle et al. (2013) that many learners have difficulty identifying the unit on the number line. Larson (1987) also points out that some learners "disregard the scaling and treat the whole number line as a unit" (p. 398) and sometimes learners seem not able to relate the number of divisions in each unit to the denominator of the fraction.

2.8 CONCLUSION

In the context of this study, fractions (common fractions) refer to a representation of rational numbers in the form a/b (Hansen, 2015). The learning of fractions is very complex and is difficult to teach, because these numbers have multiple interpretations and representations. The limited use of the part-whole model to find a fraction less than a unit by counting parts, appears to make learners think of a fraction as two unrelated whole numbers. The use of multiple models and the extensive use of a number line to find fractions, are regarded as some of the best practices that can help learners to realise that fractions are indeed numbers.

Therefore, the teaching of fractions using models is recommended to help learners develop conceptual knowledge and procedural fluency of fractions.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 INTRODUCTION

This chapter presents the research methodology I used to achieve the following research goals: a) To describe the nature of Grade 8 learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention; b) To identify and describe the changes in Grade 8 learners' conceptual understanding of and procedural fluency with fractions after the teaching intervention and; c) To describe the possible influence of the teaching intervention on Grade 8 learners' conceptual understanding of and procedural fluency with fractions. This chapter discusses the research site; sample and sampling method; research orientation; research methodology; research design; data collection methods and tools; and data analysis. Finally, the chapter discusses validity and reliability for this research; ethical considerations; and limitations and challenges for this research project.

3.2 PROFILE OF THE RESEARCH SITE

This study was conducted at one of the urban secondary schools located at the periphery of Tsumeb in Oshikoto Region, Namibia. Tsumeb is a gateway to the northern part of Namibia and it is 252km from Ondangwa and 432km from Windhoek. The school was established in 1978 and offers schooling for Grade 8-12. The school population is multicultural and of different ethnic groups comprising of Oshiwambo, Damara, Herero, Kavango, and Caprivi. There are 40 teachers at the school and 800 learners. The school has enough classrooms to accommodate all learners and teachers. It also has a hostel that accommodates 300 learners each year, while most of the learners at the school are non-boarders. The school was selected due to a positive response given by the school authority and the interest shown by Grade 8 mathematics teachers at the school, for their learners to partake in this study. In addition, the school was selected as it has a hostel and participants selected for this study were all boarding learners, which enabled them to attend classes after school hours in the afternoons. The period for conducting this research study was 3rd June - 3rdJuly 2015 during the second school trimester.

3.3 SAMPLING METHOD AND SAMPLE OF THE STUDY

A purposive sampling technique was used to select the sample for this study. Purposive sampling is when the researcher "makes specific choices about which people, groups or objects to include in the sample" (Bertram & Christiansen, 2014, p. 60). A sample of 12 Grade 8 learners was selected purposively by one of the Grade 8 mathematics teachers at the research school, as requested by myself. Five of the learners who participated in the study were boys, and seven learners were girls. The ages of participants ranged from 13 to 16 years, with 14 years being the modal age of participants. All selected participants were enrolled for Grade 8 for the first time. The selection of participants was based on the following criteria:

a) The participants were learners who volunteered to partake in the study and thereafter, written consent from both the parents and the learners were obtained;

b) Learners selected for this study were those perceived by their mathematics teacher as "more participative" and "able to communicate confidently" during the lessons of mathematics. Initially, I believed that it was important to work with learners who were more engaging, to enable me to probe and collect rich data during class discussions and one-on-one interviews with learners. This decision was based on my own classroom teaching experience – I teach mathematics to learners for whom English (the medium of instruction) is their second language and as a result, the majority of learners are shy to communicate (and participate) in class due to their English. Often, learners are less engaged in class discussions, to the extent that most of them are reluctant to ask for further explanations when they do not understand. Two of the learners (L4 and L5) showed such difficulties of communication, as expected during the intervention;

c) Only learners who were accommodated in the hostel were selected. Lessons for the intervention were conducted during the afternoon after normal school hours. It was necessary for all learners to reside in the hostel so that they could attend lessons in the afternoon (between 14:30 and 16:30) after taking their lunch meal, to avoid learners attending lessons on empty stomachs, as there were no funds reserved to buy meals for research participants.

3.4 RESEARCH ORIENTATION

Bertram and Christiansen (2014) describe a research paradigm as the representation of the worldviews that influence the approach the researcher chooses, to conduct his/her research in an acceptable way. Bertram and Christiansen (2014) indicate that the research paradigm is crucial in every social science research, because it influences the type of research questions the researcher seeks to find answers to; the methods of collecting data; the choice of unit of analysis; and how the researcher analyses data and interprets the findings. In this study, the research paradigm that informed the research design was the qualitative, interpretive paradigm.

3.4.1 Interpretive paradigm

This study falls within the interpretive paradigm. According to Bertram and Christiansen (2014), the interpretive paradigm involves multiple interpretations of the research data. In educational research, the interpretive paradigm is used to understand "the meaning which informs human behaviour" (Bertram & Christiansen, 2014, p. 26). In this study, I made use of data drawn from four research instruments, namely: semi-structured interviews, tests, learners' worksheets, and transcripts of lesson videos. According to Bertram and Christiansen (2014), this study falls within the interpretative paradigm, because its findings were drawn from spoken and written responses of research participants, and are therefore authentic since they are the research participants' reports of their actual experience (Bertram & Christiansen, 2014).

3.4.2 Qualitative research

Maree (2015) describes qualitative research as research:

> that attempts to collect rich descriptive data in respect of a particular phenomenon or context with the intention of developing an understanding of what is being observed or studied. It therefore focuses on how individuals and groups view and understand the world and construct meaning out of their experiences. (p. 50)

In this study, the paradigm of qualitative research refers to examining the interaction of research participants and "observing the participants in their natural environment (*in situ*) and focusing on their meanings and interpretations" (Maree, 2015, p. 51). According to Bertram and Christiansen (2014), qualitative data usually "consists of textual or visual data ... [like] field notes, recording observations, photographs ... [or] drawing[s]" (p. 116). In this study, the qualitative data consists of the spoken and written responses of the research participants drawn from the tests, interviews, learners' worksheets and transcripts of lesson videos. This data also includes all the drawings for the number lines and area models that learners used in this study. In qualitative research, the subjectivity of the researcher is acknowledged in describing and understanding the investigated phenomenon (Bertram & Christiansen, 2014), and the methods of analysis are informed by this consideration.

3.5 RESEARCH METHODOLOGY

The research methodology chosen for this research study is a case study.

3.5.1 Case study

A case study has multiple definitions. Bertram and Christiansen (2014) define a case study as a "systematic and in-depth study of one particular case in context, where the case may be a person (such as a teacher, a learner, a principal or parent), a group of people (such as a family or a class of learners), a school, a community, or an organization" (p. 42). Bertram and Christiansen (2014) further describe a case study as a "style of research that is often used by researchers in the interpretative paradigm" (*ibid.*) with the aim to describe the nature of a particular situation. Creswell (2013) defines a case study research as "a qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) … through detailed, in-depth data collection involving multiple sources of information (e.g. observations, audio-visual material, and documents and reports), and reports a case description and case themes" (p. 97). Besides a number of definitions of a case study, in a case study, the researcher presents "the reality of the participants' lived experiences of and thoughts about a particular situation" (Bertram & Christiansen, 2014, p. 42).

This study meets the criteria of the descriptions of a case study above, in three ways. Firstly, this research study is an in-depth study of a single case of an intervention (of 12 Grade 8 learners) whose overall teaching goal was teaching fractions in a meaningful way (learning fractions with conceptual understanding and procedural fluency) using area models and number lines as opposed to the use of traditional teaching methods to teach fractions. Secondly, the data presented in Chapter Four and Five of this research, represents the participants' lived experiences, because these are the spoken and written responses of the research participants drawn from the tests, interviews, learners' worksheets and transcripts of lesson videos. Thirdly, this research study uses multiple sources of data. Creswell (2013) states that the use of multiple sources of data is good to develop an in-depth-understanding of the case. Further, the unit of analysis of this research are the spoken and written responses of the research participants.

This study is an intrinsic case study. Creswell (2013) describes an intrinsic case study design as a qualitative case study "composed to illustrate a unique case, a case that has unusual interest in and of itself and needs to be described and detailed" (p. 98). This study is an intrinsic study, because I chose to teach and observe my own intervention. Observing my own teaching intervention, helped me to be in a position to understand and explain the possible influences of the teaching intervention, in the emergent changes in learners' conceptual understanding of and procedural fluency with fractions, after the intervention.

3.6 RESEARCH DESIGN

Creswell (2009) describes research design as "the plan and procedures for research that span the decisions from broad assumptions to detailed methods of data collection and analysis" (p. 3). This plan involves several decisions, but there is no definite sequence for taking these decisions (Creswell, 2009). The research design of this research study involved four phases of data collection, namely: Phase 0, 1, 2 and 3. The data analysis involves reporting on the entire case by giving both descriptions of a case and themes of findings that the researcher uncovered in studying the case. The following is a description of the phases of data collection.

3.6.1: Phase 0

This phase involves the design and construction of the five research tools, namely: learners' worksheets for the teaching intervention (see appendix A); two diagnostic tests (pre-test and post-test); and two interview schedules one for the pre-interview and one for the post-interview. The intervention consisted of 13 worksheets, namely: Worksheet 1, 2A, 2B, 3, 4, 5, 6A, 6B, 6C, 7, 8A, 8B and 9. In addition, some worksheets for the teaching intervention were developed drawing on published teaching resources, namely: Worksheet1-2B and 3 were adapted from two web pages (http://www.visualfractions.com and http://www.mathworksheets4kids.com); Worksheet 4 was designed by reviewing four Namibian school textbooks for Mathematics, namely: Maths for Life Grade 5 (Lategan &

Silver, 2013), Discover Mathematics Grade 6 Learner's book (Labuschagne & Marchant, 2013), Mathematics in Context Grade 7 (Van der Westhuizen et al., 2013) and y=mx+c to success Grades 8-10 (D'Emiljo, 2010); while Worksheet 5-9 were fully constructed by myself, according to research goals and using my passion for teaching creatively and with guidance from my research supervisor.

The teaching and learning content of the intervention covers four themes, namely: identifying and naming fractions, comparing fractions, ordering fractions and adding fractions. These themes are related to the worksheets as follows:

- Identifying and naming fractions includes Worksheet 1, 2A-2B, and 3-5;
- Comparing fractions includes Worksheet 6A-6C and 7;
- Ordering fractions includes Worksheet 8A and 8B; and
- Adding fractions includes Worksheet 9.

Tables 3.1 - 3.4 present the design of the content for the teaching intervention. The tables are organised according to themes. The learners' worksheet numbers and worksheet topics are the headings or subheadings of the tables. Each table presents the learning objectives and teaching objectives of a worksheet, and activities of learners and of a teacher in a worksheet.

Table 3. 1: Identifying and naming fractions – Worksheet 1 - 5

Worksheet 1: Identifying and naming fractions			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Understand the conceptual meaning of fraction symbols as representations or measures of quantities not equal to whole units. 2. Identify fractions less than one whole unit using both circles and number lines. 3. Name fractions less than one whole unit using appropriate names, e.g. "two thirds" for 2/3. 4. Recognise a fraction	1. Explain the conceptual meaning of fraction symbols as representations or measures of quantities not equal to whole units using circles and number lines. 2. Demonstrate how to identify fractions less than one whole unit using circles and number lines. 3. Demonstrate how to read fractions using appropriate names. 4. Demonstrate the	1. Complete Worksheet 1 as per instruction and ask for guidance from the classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. Explain the conceptual meaning of fraction symbols. 2. By using the circles and number lines, demonstrate how to identify fractions less than one whole unit and how to read fractions using appropriate names. 2. By facilitating, afford learners the opportunity to complete Worksheet 1. 3. Give feedback to the class.

denominator as the number of unit subdivisions.	relationship between the whole unit, unit subdivisions and the denominator of fractions using both circles and the number lines		4. Consolidate the lesson by reinforcing the importance of reading fractions using appropriate names and the conceptual meaning of a denominator of fractions.
Worksheet 2A: Identifying fractions greater than one whole unit, less than one whole unit and equal to one whole unit using number lines then circles			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Identify fractions greater than one whole unit, less than one unit and equal to one whole unit using number lines; and express fractions greater than one whole unit using both improper fraction notation and mixed fraction notation. 2. Represent the fractions identified in no. 1 above as area shaded using circles.	1. Demonstrate how to identify fractions greater than one whole unit using the number lines and express the fractions identified using both improper fraction notation and mixed fraction notation. 2. Demonstrate how to represent the fractions identified in no. 1 above as area shaded using circles. Learners should be told to use the unit subdivisions of the number lines or denominator of fractions identified to partition the circles appropriately.	1. Complete Worksheet 2A individually and ask for help from classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. By using number lines, demonstrate how to identify fractions greater than one unit using both improper fraction notation and mixed fraction notation. 2. Demonstrate how to use a denominator or unit subdivisions to represent the fractions identified using circles. 3. By facilitating, afford learners the opportunity to complete Worksheet 2A and probe learners to explain how they used number lines to identify fractions and represent fractions identified using circles. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing on how to use unit subdivisions to identify a denominator or the whole unit using number lines and circles.
Worksheet 2B: Identifying fractions greater than one whole unit using circles then number lines			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**

1. Identify fractions greater than one whole unit using circles and express fractions identified using both improper fraction notation and mixed fraction notation. 2. Represent fractions identified in no1 above as area shaded using number lines. 3. Describe the process of expressing fractions using improper fraction notation and mixed fraction notation. 4. Explain when a fraction identified can be expressed using the improper fraction notation and mixed fraction notation.	1. Demonstrate how to identify fractions greater than one whole unit using circles and express fractions identified using both improper fraction notation and mixed fraction notation. 2. Demonstrate how to represent fractions identified in no. 1 above as area shaded using number lines.	1. Complete Worksheet 2B individually and ask for help from the classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. By using the circles, demonstrate how to identify fractions greater than one unit using both improper fraction notation and mixed fraction notation. 2. Demonstrate how to use the denominator or unit subdivisions to represent the fractions identified using the number lines. 3. By facilitating, afford learners the opportunity to complete Worksheet 2B and probe learners to explain how they used the circles to identify the fractions and represent fractions identified using number lines. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing the properties of improper fractions and mixed fractions for being representations for all fractions greater than one whole unit of the number lines and circles.
Worksheet 3: Identifying fractions less than one whole unit using number lines			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Identify fractions less than one whole unit using number lines. 2. List all common properties of the fractions identified in no. 1 above. 3. Recognise proper fractions as fractions less than one whole unit.	1. Ask the class to identify fractions at the marked position of the number lines.	1. Complete Worksheet 3 individually and ask for help from classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. By facilitating, afford learners the opportunity to complete Worksheet 3 and probe learners to explain how they used number lines to identify the fractions. 2. Ask the class to identify the fractions at the marked position of number lines and

			using the experience they acquired from previous worksheets. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing the properties of proper fractions for being representations for all fractions less than one whole unit of the number lines and circles.
Worksheet 4: Locating fractions less than one whole unit and fractions greater than one whole unit using number lines			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Locate fractions less than one whole unit and fractions greater than one whole unit using the number lines. Learners should be reminded to use a denominator of given fractions to partition whole units and use a numerator to locate a fraction appropriately.	1. Demonstrate how to use a denominator of given fractions to partition whole units of number lines and locate fractions on the number lines using the numerators appropriately. 2. The teacher should help learners to locate fractions greater than one whole unit easily by suggesting learners to express the given fractions in the mixed fraction notation in order to determine the minimum number of whole units to show on the number lines.	1. Complete Worksheet 4 individually and ask for help from classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. By using number lines, demonstrate how to use a denominator of given fractions to partition whole units of number lines and locate fractions on the number lines using numerators appropriately. 2. By facilitating, afford learners the opportunity to complete Worksheet 4 and probe learners to explain how they used the given fractions to locate the fractions on the number lines. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing the conversion of fractions greater than one whole unit into the mixed fraction notation as the quicker strategy for determining the minimum number of whole units needed to represent fractions.
Worksheet 5: Locating fractions less than one whole unit and fractions greater than one whole unit using number lines			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**

1. Locate fractions less than one whole unit and fractions greater than one whole unit using pre-partitioned number lines.	1. Ask the class to locate the given fractions using pre-partitioned number lines appropriately.	1. The activities for the learners here were the same as those of Worksheet 4.	The activities for the teacher here were the same as those of Worksheet 4. Finally, the teacher should reinforce the difference between mixed fractions (and improper fractions) as fractions greater than one whole unit, and proper fractions as fractions less than one whole unit.

Table 3. 2: Comparing fractions – Worksheet 6A – 7

Worksheet 6A: Comparing fractions with same numerators using fraction bars			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Shade equal pre-partitioned fraction bars to compare two fractions with same numerators –*see Worksheet 6A* 2. Describe what is special about comparing fractions in Worksheet 6A.	1. Demonstrate how to compare pairs of fractions with same numerators appropriately using equal pre-partitioned fraction bars based on the sizes of area shaded of the equal fraction bars. The teacher should tell learners that the inequality signs (<, =, >) always face the larger number.	1. Complete Worksheet 6A individually and ask for help from the classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session. Learners should first complete this worksheet before attempting to complete Worksheet 6B.	1. By using equal pre-partitioned fraction bars, demonstrate how to compare fractions with same numerators according to the sizes of the area shaded of the equal fraction bars. 2. By facilitating, afford learners the opportunity to complete worksheet 6A and probe learners to explain how they used the sizes of the area shaded of equal fraction bars to determine the bigger fraction. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing the rules for comparing fractions with the same numerators.
Worksheet 6B: Comparing fractions with same denominators using fraction bars			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Shade equal pre-partitioned fraction bars to compare fractions with same denominators –	1. Demonstrate how to compare pairs of fractions with same denominators appropriately using equal pre-partitioned fraction bars based	1. Complete Worksheet 6B individually and ask for help from the classmates and teacher if necessary. 2. Share their	1. By using equal pre-partitioned fraction bars, demonstrate how to compare fractions with same denominators according to the sizes of area shaded of equal fraction bars.

see Worksheet 6B. 2. Describe what is special about comparing fractions in Worksheet 6B.	on the sizes of area shaded of the equal fraction bars. The teacher should tell the class that the inequality signs (<, =, >) always face the larger number.	answers with the class and make corrections during the feedback session. Learners should first complete Worksheet 6B before attempting to complete Worksheet 6C.	2. By facilitating, afford learners the opportunity to complete Worksheet 6B, and probe learners to explain how they used the sizes of area shaded of equal fraction bars to determine the bigger fraction. 3. Give feedback to the class. 4. Consolidate the lesson by reinforcing the rules for comparing fractions with the same denominators.

Worksheet 6C: Comparing two fractions with different numerators and denominators using fraction bars

Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Shade equal pre-partitioned fraction bars to compare fractions with different numerators and denominators *–see Worksheet 6C* 2. Describe what is special about comparing fractions in Worksheet 6C.	1. Demonstrate how to compare fractions with different numerators and denominators appropriately using equal pre-partitioned fraction bars based on the sizes of area shaded of equal fraction bars. The teacher should tell the class that the inequality signs (<, =, >) always face the larger number.	1. Complete Worksheet 6C individually and ask for help from the classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	Same activities as those of Worksheet 6B applies here.

Worksheet 7: Comparing fractions using number lines

Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Locate the given fractions on the number lines and use the positions of the fractions to compare fractions with same denominators, fractions with same numerators and fractions with different numerators and denominators appropriately. 2. Use benchmarking to compare fractions appropriately, e.g.	1. Demonstrate how to use benchmarking to compare fractions appropriately.	1. Complete Worksheet 7 individually and ask for help from the classmates and teacher if necessary. 2. Share their answers with the class and make corrections during the feedback session.	1. By using equal number lines, demonstrate how to compare two fractions according to the positions of the fractions on the number lines. 2. By facilitating, afford learners the opportunity to complete Worksheet 7 and probe learners to explain how they used number lines and benchmarking to determine the bigger fraction. 3. Give feedback to the class.

learners should recognise that the farthest fraction from 0 is always the bigger one and the fraction closer to 0 is the smallest.			4. Consolidate the lesson by reinforcing the use of benchmarking for comparing fractions such that the fraction closer to 0 is the smaller one and the fraction farthest from 0 but closer to 1 is the biggest.

Table 3. 3: Ordering fractions – Worksheet 8A – 8B

Worksheet 8A: Ordering three fractions using fraction bars			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Compare and order three given fractions from the smallest fraction to the largest fraction using benchmarking and sizes of the area shaded of equal, pre-partitioned fraction bars.	1. Demonstrate how to compare and order three given fractions from the smallest fraction to the largest fraction using benchmarking and sizes of the area shaded of equal, pre-partitioned fraction bars.	The activities for the learners here were the same as those of Worksheet 7.	The activities for the teacher here were the same as those of Worksheet 7.
Worksheet 8B: Ordering three fractions using number lines			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Compare and order three given fractions from the smallest fraction to the largest fraction using benchmarking and according to the positions of fractions on equal number lines appropriately.	1. Demonstrate how to order three fractions from smallest to largest marked positions representing fractions on number lines.	1. The activities for the learners here were the same as those of Worksheet 7.	1. The activities for the teacher here were the same as those of Worksheet 7.

Table 3. 4: Adding fractions – Worksheet 9

Worksheet 9: Adding two fractions using fraction bars			
Learning/lesson objectives: ***It was expected of the learners to:***	**Teaching objectives:** ***It was expected of a teacher to:***	**Learners' activities**	**Teacher's activities**
1. Use the lowest common denominator method to find the sum of two fractions with the same denominator and two fractions with different denominators appropriately. 2. Use the fraction bars to represent visually the lowest	1. Demonstrate how to use the lowest common denominator method to find the sum of two fractions with the same denominator and two fractions with different denominators	1. The activities for the learners here were the same as those of Worksheet 7.	1. The activities for the teacher here were the same as those of Worksheet 7. 2. The teacher should reinforce to the class the

common denominator method for adding fractions. Learners should recognize that only equally sized units are countable by repartitioning the different sized parts of the fraction bars.	appropriately. 2. Demonstrate how to use the fraction bars to represent visually the lowest common denominator method for adding fractions.		importance of using the lowest common denominator method which is making equally sized units for them to become countable.

After construction of the worksheets for the teaching intervention, I designed two diagnostic tests and two interview schedules according to the themes of the teaching intervention.

3.6.2: Phase 1

This phase of data collection for the research study involved administering both the pre-test and pre-interview to the participants respectively. The duration of this phase was two consecutive days. Firstly, all 12 participants wrote the pre-test (1 hour and 30 minutes long) on the 9th June 2015, which was the day after all the participants gave their consent. The pre-test scripts were skimmed over and four learners were chosen for a pre-interview according to their responses. The four learners who were interviewed for a pre-interview, were two learners that showed very limited conceptual understanding of and procedural fluency with fractions, as most of their responses were incorrect and this criteria was decided on by myself, before data collection. These two learners were L3 and L5. The other two learners (L6 and L7) who participated in the pre-interview, showed higher conceptual understanding of and procedural fluency with fractions, as most of their responses were correct. The pre-interviews were all one-on-one interviews and administered on the 10th June 2015. Each interview lasted for about 25 minutes. Both pre-tests and pre-interviews took place in the afternoons, between 14:30 and 16:30.

During the pre-interviews, learners were given an interview questionnaire similar to my own and were asked to answer all the questions by writing down the answers in the spaces provided in the questionnaire. Learners were probed during the interviews to explain how they arrived at their answers. The general observation made during the pre-interviews, showed that all learners interviewed displayed the same understanding of fractions, similar to their responses in their respective pre-tests. The interviews were all video recorded using a video camera supported by a camera stand. The video recording of interviews was preferred by the researcher to enable him to document the thinking process of how every learner arrived at their answers by looking carefully at what a learner was writing (e.g. during the pre-interviews, initially all learners drew a number line using whole numbers to label unit subdivisions instead of using the appropriate fraction symbols, and after they were probed whether their labelling were appropriate, only L4 then changed his labelling of unit subdivisions using appropriate fraction symbols) . The pre-test scripts and videos of the pre-interviews were filed for detailed data analysis, after the data collection process.

3.6.3: Phase 2

This phase of data collection involved the teaching of the intervention, using learners' worksheets. A series of 13 worksheets as described in Table 3.1 - 3.4 were used to teach fractions for a period of nine days between 11th June 2015 and 30th June 2015. Table 3.5 below shows the dates for conducting the nine lessons of the intervention and the number of worksheets used per lesson. The worksheets were completed individually by 12 participants with the help of the classmates and a teacher, where necessary. Learners were encouraged to show all workings, including explanations of procedures used, as they completed the worksheets. The role of the teacher during the lessons was to probe learners and through this approach, guide the learners to make use of appropriate concepts and procedures of fractions. All lessons were recorded using a video camera for later transcription. The completed worksheets of learners were collected and filed for data analysis.

Table 3.5: The dates of the activities conducted during the process of data collection

Dates	03.6.15	09.6.15	10.6.15	11.6.15	13.6.15	15.6.15	16.6.15	19.6.15	22.6.15	23.6.15	24.6.15	30.6.15	01.7.15	02.7.15	03.7.15
Activities for data collection	Selection of participants	Pre-test	Pre-interviews	**Lessons**									Post-test	Post-interviews	Recall interviews
				1	2	3	4	5	6	7	8	9			
Worksheet number used per lesson				1	2A 2B	3	4, 5	6A 6B 6C	7	8A	8B	9			

3.6.4: Phase 3

This was the last phase of data collection of the research study. This phase involved administering the post-intervention test (post-test), post-intervention interviews (post-interviews) and recall interviews respectively. Table 3.5 above shows the dates for administering the post-test, post-interviews and recall interviews. The post-test was administered to all 12 participants. The post-interview was administered to four learners and these learners were the same learners that were interviewed for the pre-interviews. The post-interviews were all one-on-one interviews.

During the post-interviews, learners were given an interview questionnaire similar to my own and were asked to answer all the questions by writing down the answers in the spaces provided of the questionnaire. Learners were probed during the interview to explain how they arrived at the answers. The general observation made during the post-interviews showed that all learners interviewed showed the same understanding of fractions similar to their written responses in their respective post-tests.

The recall-interviews were administered to all 12 participants, but as one-on-one interviews. These interviews were unstructured interviews, administered by simply asking the learners to give a general reflection of their experience of the intervention, by explaining the most salient things they learned from the intervention. All interviews were recorded using the video-camera supported by a camera stand and filed for transcription. The post-test scripts were also collected and filed for detailed data analysis.

3.7 DATA COLLECTION METHODS

This research study made use of four methods of data collection, namely: testing, interviews and observation and learners' written work. Each method is discussed in detail in the following sections.

3.7.1 Testing

Bertram and Christiansen (2014) describe testing in the interpretative paradigm, as a method of data collection using tests with open-ended questions. Bertram and Christiansen (2014) explain the advantage of using open-ended questions in a test as they "leave more space for unpacking the thinking of the respondents, but also take longer to code" (p. 96). They explain that in testing, the researcher needs to "test learners before the intervention (the pre-test) to ascertain their achievement … and then test them again after the intervention (the post-test)" (Bertram & Christiansen, 2014, p. 95). In this research study, I made use of the testing method by administering the pre-intervention test and post-intervention test (see Appendices B and C) to all 12 participants. This enabled me to identify possible changes in the learners' conceptual understanding of and procedural fluency with fractions, as well as possible influences of the intervention.

Bertram and Christiansen (2014) define a pre-test as a test administered before the intervention and a post-test as a test administered after the intervention. In this research study, a pre-test refers to a diagnostic test written by research participants before the teaching intervention. Put differently, a pre-test means the pre-intervention test. In this study, a post-test refers to a diagnostic test written by research participants, after the teaching intervention. In other words, a post-test refers to a post-intervention test.

Bertram and Christiansen (2014) describe a test as a straightforward method of determining whether learners have mastered the content, even though "a test which simply shows all participants doing very well or very badly is unlikely to tell the researcher much about the spread of the participants' knowledge and ability" (p. 95). The analysis of tests in this study showed a number of changes in learners' conceptual understanding of and procedural fluency with fractions, as well as possible influences of the intervention. The analysis of tests also showed that some learners benefited more from the intervention than others and thus two groups of learners according to the changes in their responses in the pre-test and post-test, were identified and presented as Group A and group B in Chapter Four (see section 4.2).

The pre-test and post-test contained multiple choice questions and open-ended questions. Both tests had similar structures and assessed learners on four themes of fractions, namely: fraction concepts; comparing two fractions; ordering three fractions; and adding two fractions. *Fraction concepts* involved identifying and representing fractions less than one whole unit and fractions greater than one whole unit, using the number lines and circles. *Comparing two fractions and ordering three fractions* involved comparing the given fractions, using procedures chosen by the learners and showing the workings or explanations of how they used the chosen procedures to compare and order fractions. *Adding two fractions* involved adding two fractions with the same denominator and fractions with different denominators, respectively. Learners were also expected to represent visually the lowest common denominator method for adding fractions, using the fraction bars.

Bertram and Christiansen (2014) indicate that when testing is used in the interpretivist paradigm, which is the paradigm used for this research, other methods of data collection such as open-ended questions should be used as well to validate the data, hence the use of interviews, observation and learners' written tasks, as methods for data collection in this research.

3.7.2 Interviews

An interveiw is "a conversation between the researcher and the respondent" (Bertram & Christiansen, 2014, p. 80). Bertram and Christiansen (2014) describe the difference between the interview and every day conversation in the sense that for an interview, the researcher is the one who sets the agenda and asks questions; that an interview is mostly a structured and focused conversation where the researcher keeps in mind "particular information that he or she wants from the respondent, and has designed particular questions to be answered" (p. 80).

In this research study, I made use of both structured and unstructured interviews. A structured interview is when "the researcher uses an interview schedule, which is a set of questions in a predetermined order. In the structured interview, the questions may require closed responses … or open-ended responses" (*ibid.*). An unstructured interview is "when the researcher may simply introduce the topic or main research question, then let the respondent answer in the way that he or she would like to" (*ibid.*).

The pre-interviews and post-interviews for this research were both structured interviews (see Appendix D and E).The questionnaires for the pre-interviews and post-interviews were similar to each other and similar to the structure of tests since they were all organised according to the same themes. Four learners (L3, L5, L6 and L7) were interviewed for both interviews.

The recall interviews were all unstructured interviews, administered after the writing of the post-test and post-interviews, by simply asking every learner to give a verbal reflection of their experience with the intervention, and recollect the most salient things they learned from the intervention. The purpose of recall interviews was to collect primary data from the participants about their experiences derived from the teaching intervention and use this data to support the findings for changes in learners' conceptual understanding of and procedural fluency with fractions, as well as the possible influence of the intervention. All 12 learners were interviewed individually for the recall interviews, after writing the post-test. All three interviews were video-recorded and transcribed (see Appendices F, G and H).

3.7.3 Observation

The third method used for data collection was observation. Observation refers to "the systematic process of recording the behavioural patterns of participants, objects and occurrences without necessarily questioning or communicating with them" (Maree, 2015, pp. 83-84). The observation of this research study involved video-recording the lessons, using a video camera, by a friend. The role of the researcher during observation was that of a "participant as observer".

Maree (2015) describes a participant as observer as:

> the researcher [who] becomes a participant in the situation being observed, and may intervene in the dynamics of the situation and even try to alter it. The researcher thus immerses himself or herself in a chosen setting to gain an insider perspective (called an emic perspective) of that setting. (p. 85)

My role in this study was as a participant in the research process by teaching the intervention of fractions to the participating learners. The objective I hoped to achieve by teaching the intervention myself, was to develop learners' conceptual understanding of and procedural fluency with fractions using number lines and area models. By doing this, it allowed me to observe and understand the way learners were thinking about fractions before the intervention, and the process that contributed to the changes in learners' conceptual understanding of and procedural fluency with fractions, as a result of the intervention. The observational data was used to discuss the possible influence of the teaching intervention as presented in Chapter Five.

3.7.4 Documents: Learners' worksheets

The fourth method of data collection used in this study was documents. Documents refer to "all types of written communication that may shed light on the phenomenon" being investigated (Maree, 2015, p. 82). Maree (2015) states that documents may include published and unpublished documents such as company reports, letters, email messages, or any document that is connected to the investigation. In this research study, documents refer to the learners' worksheets that they completed during the intervention. The completed worksheets for the learners were collected and filed for data analysis. The worksheets of learners were the same worksheets used to teach the intervention. These worksheets served as primary sources of data, since they contained unpublished data gathered directly from the participants by myself.

3.8 DATA ANALYSIS

This research used an inductive analysis approach to analyse the data. Thomas (2006) describes inductive analysis as the "approaches that primarily use detailed readings of raw data to derive concepts, themes, or a model through interpretations made from the raw data by an evaluator or researcher" (p. 238). In this study, inductive analysis refers to the approach I adopted that used detailed readings of raw data to develop inductive codes and grouping of the codes into themes. Inductive codes refer to the codes that I derived from the raw data. (Maree, 2015).

The analysis of data in this study involved five phases (Phase 1-5) which are discussed below. The analysis of raw data started by sorting data from the pre-test and post-test according to organisational categories, using frequency tables. Maxwell (1998) describes organisational categories as generally broad subjects or issues established prior to the conducting of interviews and observation. In this study, organisational categories refer to the themes of the teaching interventions, tests and interviews. Table 3.6 below shows how these organisational categories and inductive codes were used to analyse data from the tests and interviews.

Table 3.6: Phases for the process of data analysis

Data analysis phases		
Phase 1	**Phase 2**	**Phase 3**
Organisational categories	**Inductive codes**	**Inductive codes**
Identifying and naming fractions	Appropriate fraction notation and fraction names	Appropriate relating of unit subdivisions to the denominator of fractions
	Inappropriate fraction	Inappropriate relating of unit subdivisions to the

<table>
<tr><td></td><td>notation and fraction names</td><td>denominator of fractions</td></tr>
<tr><td rowspan="2">Comparing two fractions</td><td>Appropriate procedures</td><td rowspan="2">Appropriate procedures for comparing and ordering fractions, e.g. benchmarking and rules for comparing fractions</td></tr>
<tr><td>Inappropriate procedures</td></tr>
<tr><td rowspan="2">Ordering three fractions</td><td>Appropriate procedures</td><td rowspan="2">Inappropriate procedures for comparing and ordering fractions, e.g. using the sizes of denominators only</td></tr>
<tr><td>Inappropriate procedures</td></tr>
<tr><td rowspan="4">Adding two fractions</td><td rowspan="2">Appropriate procedures</td><td>Using the lowest common denominator methods appropriately</td></tr>
<tr><td>Using fraction bars to represent visually the lowest common denominator method appropriately</td></tr>
<tr><td rowspan="2">Inappropriate procedures</td><td>Using the lowest common denominator methods inappropriately</td></tr>
<tr><td>Using fraction bars to represent visually the lowest common denominator method inappropriately</td></tr>
</table>

The analysis of data in this research study was a process of three phases as shown in Table 3.6. In the first phase (Phase 1), I used the organisational categories to categorise the data, drawn from the tests and interviews, into four categories. The second phase of analysis involved identifying and classifying the learners' responses for each organisational category into two subcategories, namely: appropriate and inappropriate responses. The coding of data during the second phase of analysis was done using the frequency tables (see Tables 4.1- 4.8 in Chapter Four). The responses that were inappropriate are highlighted in grey, in the frequency tables in Chapter Four. The frequency tables allowed me to see the emerging patterns from the data of the nature of learners' conceptual understanding of and procedural fluency with fractions before and after the intervention. The frequency tables also helped me to draw the links between similar codes. The linking of similar codes helped me to develop substantive, inductive codes for Phase 3 of the analysis. The inductive codes of the third phase were turned into emergent categories from the data. These emergent categories are presented in Chapter Four as the headings for the subsections of Sections 4.3 and 4.4. The subsections of Chapter Four present the findings for the first and second research questions of this research project.

The fourth phase of analysis which is not listed in Table 3.6, involved the analysis of the transcripts of the lesson videos (data of observation) and learners' worksheets. The analysis of learners' worksheets and transcripts of the lesson videos were essential to provide explanations for the changes presented in Section 4.4 of Chapter Four. Put differently, the analysis of learners' worksheets and transcripts of the lesson videos were important in finding answers for the third research question of this study. The changes presented in Section 4.4 of

Chapter Four were used to code data in the transcripts of the video lessons. The coding of data in the transcripts for the video lessons led to five emerging categories for the influence of the teaching intervention, namely: reading fractions, using appropriate names; developing the conceptual understanding of the denominator of fractions by writing whole units as fractions of unit subdivisions; differentiating fractions less than one whole unit from fractions greater than one whole unit; using benchmarking and rules for comparing and ordering fractions; and representing visually the lowest common denominator method for adding two fractions. The emergent categories were coded in the transcripts of interviews and lesson videos using five colours. Table 3.7 below shows the colours used to code the data. These colours enabled the researcher to identify quotes and extracts of learners' responses (taken from Appendices F-I) which are presented in Chapter Four and Five.

Table 3. 7: The inductive codes of emergent categories for the fourth phase of data analysis

Emergent categories	**Color used for coding the categories**	**Color**
Reading fractions		Light green
Developing the conceptual meaning of the denominator of fractions by writing whole units as fractions of unit subdivisions		Light blue
Differentiating fractions less than one whole unit from fractions greater than one whole unit		Purple
Using benchmarking and rules for comparing and ordering fractions		Pink
Representing visually the lowest common denominator method for adding two fractions		Red

Finally, the fifth phase of the process of data analysis involved formulating the themes of the research findings which are presented in Chapter Six. The themes of the findings of this research project were formulated by relating the categories presented in Chapter Four and Five to answer the research questions. In Chapter Four and Five, data is presented in the form of frequency tables, extracts from learners' responses and snapshots of learners' workings. The snapshots of learners' working are presented as figures (see Figure 4.1 - 4.22 and 5.1 - 5.12 in Chapters Four and Five respectively).

3.9 VALIDITY AND RELIABILITY FOR THIS RESEARCH PROJECT

Validity of qualitative research designs refer to "the degree to which the interpretations and concepts used have mutual meaning both for the participants and researcher" (Maree, 2015,

p. 38). This research study maintained the internal validity of its findings through crystallisation. Crystallisation refers to the "practice of 'validating' results by using multiple methods of data collection and analysis" (Maree, 2015, p. 40). In this research study, four methods of data collection were used, namely: testing, interviews, observation and documents. Dempsey (2010) emphasises that the use of multiple data sources is crucial for ensuring the internal validity of the qualitative (interpretive) research. Maree (2015) also stresses that the credibility of the research findings can be enhanced by allowing people who have specific interest in the research to comment on or assess the research findings, interpretations and conclusions. The research findings of this research are credible, because the research supervisor for this research study has been guiding me, by assessing the validity of the research instruments, findings and conclusions.

Maree (2015) describes reliability in regard to qualitative research, as when the findings are consistent with the data collected (Maree, 2015, p. 38). The findings of this research study seem to be reliable, since there was correlation of learners' responses in the tests, to their responses in the interviews. For instance, in both pre-test and pre-interviews, the analysis of learners' responses indicated that the research participants did not relate the denominator of fraction notation to the unit subdivisions (see Subsection 4.3.1 in Chapter Four).

3.10 ETHICAL CONSIDERATIONS FOR THIS RESEARCH PROJECT

This research study respected the ethics for conducting research in a number of ways. Firstly, I obtained the written consent from the Regional Director of Education and school headmaster before conducting this research (see Appendices L and N). Secondly, I obtained written consent from both the research participants and guardians of the participants, since the participating learners were minors (see Appendices O and P). Thirdly, I also explained the objectives of the study and possible contribution of the research to the research participants, staff and the school headmaster. Fourthly, the confidentiality and anonymity of the research participants was warranted by using fictitious names (e.g. L1, L2) for identifying the research participants, and by ensuring that only I had access to the video recordings of the lessons. In addition, learners were requested to partake in this study out of their own free will and had the choice to withdraw from the research if they wished to, at any time. Lastly, I also avoided being judgmental towards the research school culture or being disruptive of the school operational policies, by acting professionally at all times.

3.11 LIMITATIONS AND CHALLENGES FOR THIS RESEARCH PROJECT

Maree (2015) explains that “the goal of qualitative research is not to generalise findings across a population, [but] … to provide understanding from the participants’ perspective” (p. 115). Since this research project was exploratory research then, its findings should not be generalised across all Grade 8 learners in Namibia, as its sample was limited to 12 learners from one selected secondary school in the Oshikoto region only; but its findings can provide understanding from the participants’ perspective and be applicable to classroom situations where teachers seek to improve their teaching practices on fractions.

The findings of this research project are limited to the observations, perception and interpretations of a novice researcher, myself, who also designed and implemented the research instruments. However, I have capitalised on my teaching experience of mathematics, the teaching resources that were at my disposal, as well as the guidance received from my research supervisor to increase the trustworthiness of the research design and research instruments according to the research goals of this study.

3.12 CONCLUSION

This part concludes the methodology chapter. This chapter described the methodology used in this study and a number of key ideas were discussed, namely: the research site; sample and sampling method; the research orientation; the research design with four phases 0-3; the methods used for data collection; the techniques and process for data analysis; issues pertaining to validity and reliability; ethical considerations; and limitations and challenges for this study which were thoroughly discussed.

CHAPTER 4

DATA PRESENTATION PART 1

4.1 INTRODUCTION

This chapter presents the findings of the research project. The data presented in this chapter are drawn from five research instruments (tests and interviews) which are: the pre-test, pre-interviews, post-test, post-interviews and recall interviews. The findings are presented in three key sections, namely: sections 4.2, 4.3 and 4.4. Section 4.2 categorises the learners into two groups, based on their responses to the pre- and post-tests. Learners in the first group displayed changes in response to some questions, but little or no changes in response to other questions, while learners in the second group displayed changes in response to most questions. A brief summary of learners' changes in conceptual understanding of, and procedural fluency with, fractions is also presented. Section 4.3 discusses the learners' conceptual understanding of, and procedural fluency with, fractions before the teaching intervention, drawing on the pre-test and pre-interviews that were administered before the intervention, as well as the recall interviews that took place after the teaching intervention. Finally, section 4.4 discusses the changes in learners' conceptual understanding of, and procedural fluency with, fractions over the period of the teaching intervention, drawing on the post-test, post-interviews and recall interviews that were administered after the intervention.

4.2 GROUPING LEARNERS ACCORDING TO THEIR RESPONSES

Learners were to be divided into two groups, based on their prior understanding before the intervention and the progress they showed after the intervention. The groups were identified through the analysis of learners' responses in the pre-test, post-test and interviews. Learners in group A, showed minimal positive change over the period of the intervention, while learners in group B showed substantial positive change over this period. Table 4.1 below presents the two groups according to the analysis of their responses, providing a summary of the changes in learners' conceptual understanding of and procedural fluency with fractions.

Table 4.1: The summary of changes in learners' conceptual understanding and procedural fluency in fractions as per three key sections of the pre-test, post-test and interviews

The summary of the changes in learners' conceptual understanding of and procedural fluency with fractions according to the three sections of a pre-test, post-test and interviews:				
Group name	**Learner**	**Three key sections for the tests and interviews**		
		Identifying and representing fractions	Comparing and ordering fractions	Adding fractions
Group A	L2	√	√	No change conceptually. Improved on adding fractions with different denominators but gets worse on adding fractions with the same denominators
	L3	√	√	No change. Added fractions with the same denominators appropriately before and after the intervention
	L4	√ fractions<1	√ fractions<1	No change conceptually. Changed from adding all types of fractions appropriately to adding fractions with different denominators inappropriately
	L5	√ fractions<1	No change. Appropriate comparison of fractions with same denominators before and after the intervention	No change (generally incorrect)
	L10	√	√	No change. Only added fractions with the same denominators appropriately after the intervention
Group B	L1	√	√	√ conceptually
	L6	√	√ conceptually	√ conceptually
	L7	√	√ conceptually	√ conceptually
	L8	√	√	√ conceptually
	L9	√	√	√
	L11	√	√	√
	L12	√	√ conceptually	√*

Keys to symbols used in table 4.1:

√: The learner showed a positive change over the period of the intervention.

√*: The learner showed stable conceptual understanding and procedural fluency for adding fractions before and after the intervention.

√conceptually: The learner showed a developing conceptual understanding of procedures after the teaching intervention. The learner used these procedures before the intervention without demonstrating the conceptual understanding.

√fractions<1: The learners showed positive change over the period of the intervention for fractions less than one whole unit only.

Gray shading indicates that learners used inappropriate procedures after the teaching intervention.

4.2.1 Group A: Learners showing minimal positive change

Five out of the 12 learners belonged to this group – learners L2, L3, L4, L5 and L10. This group of learners showed repeated use of inappropriate conceptions of fractions in the pre-test. These included the use of inappropriate fraction notation (and reading fractions using inappropriate names), as a result of not relating unit subdivisions to the denominators for both fractions less than one whole unit (proper fractions) and fractions greater than one whole unit (improper fractions and mixed fractions); expressing fractions as a count of shaded parts only; and using inappropriate procedures to compare and order fractions, including: a) using the sizes of the denominators only; b) using the numerators as divisors and denominators as dividends; and using inappropriate arithmetic manipulation (IARM) to equate fractions that are not equal.

These learners showed little positive change over the period of the intervention. A number of minor positive changes were noticed, as well as some negative changes. Firstly, the post-intervention results indicated that these learners showed a developing conceptual understanding of fractions as they related unit subdivisions to the denominator for fractions less than one whole unit (for all learners) and fractions greater than one whole unit (for L2, L3 and L10 only), as illustrated by the use of the key √ in Table 4.1. Secondly, after the intervention, four learners (L2, L3, L4 and L10) in this group showed a developing conceptual understanding and procedural fluency for comparing and ordering fractions less than one unit, using the first three appropriate procedures identified and discussed in subsection 4.4.2.1. However, after the intervention, only three learners (L2, L3 and L10) showed a developing conceptual understanding and procedural fluency for comparing and ordering fractions greater than one whole unit (refer to the use of keys √ and √*fractions<1* in Table 4.1 above). L5 did not show any change in conceptual understanding and procedural fluency for comparing and ordering fractions, although he compared and ordered fractions with the same denominators appropriately before and after the teaching intervention.

Lastly, after the intervention, no learners in this group showed *appropriate change* of conceptual understanding and procedural fluency for adding fractions. For instance, L2 added fractions with different denominators appropriately, but did not add fractions with the same denominators appropriately; while L3 and L10 added fractions with the same denominators appropriately, but added fractions with different denominators inappropriately. L5 did not show any change on the mastery for adding fractions after the intervention, as he continued to add fractions by adding numerators together and denominators together.

4.2.2 Group B: Learners showing substantial positive change

Seven learners belonged to this group – learners L1, L6, L7, L8, L9, L11 and L12. The pre-intervention results indicated that initially, the majority of the learners (L1, L6, L7 and L12) in this group used appropriate fraction notation for fractions less than one whole unit, but only two learners (L6 and L7) used appropriate fraction notation for fractions greater than one whole unit.

In the pre-test, these learners used two inappropriate procedures and three appropriate procedures, to compare and order fractions. The first inappropriate procedure involved using the sizes of the denominators only. The second inappropriate procedure involved using inappropriate arithmetic manipulation (IARM) to equate the given fractions, even when they are not equal. In contrast, the first appropriate procedure involved converting the given fractions to fractions with a common denominator. The second appropriate procedure involved using unequal fraction bars to represent the given fractions as the area shaded. The third appropriate procedure involved using rules for comparing fractions with same numerators and fractions with the same denominators (see subsection 4.3.2).

The pre-intervention results indicated that five learners (L1, L6, L7, L8 and L12) in this group, added fractions with same denominators and fractions with different denominators using the lowest common denominator method appropriately, but these learners (except L12) did not show conceptual understanding of this procedure by using fraction bars to represent this process. In the pre-test, L12 demonstrated conceptual understanding for the lowest common denominator method, by using repartitioning and shading of equal fraction bars. The pre-test results for L9 and L11 did not show the use of conceptual understanding or procedural fluency for adding fractions, as these learners used an inappropriate procedure which involved adding fractions with a new common denominator and unchanged numerators.

The post-intervention results indicated that these learners' responses changed positively in a number of ways, over the period of the intervention. Firstly, these learners showed developing conceptual understanding of fractions as they related unit subdivisions to the denominator for both fractions less than one whole unit and fractions greater than one whole unit, which was represented by the area shaded on the number line or the area model. Secondly, after the intervention, all seven learners showed developing conceptual understanding and procedural fluency for comparing and ordering fractions, and adding fractions, by using appropriate procedures (see subsections 4.4.2.1 and 4.4.3.1).

In sections 4.3 and 4.4, the findings are presented according to the progression of these two groups. This will facilitate the investigation of the possible influence of the teaching intervention.

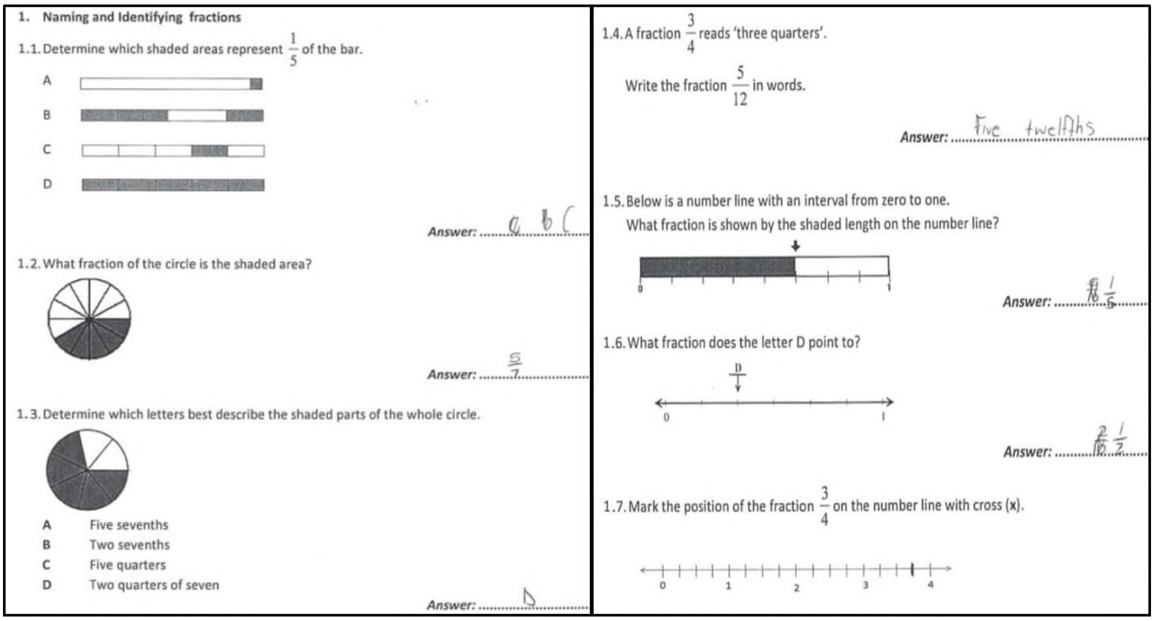
4.3 BEFORE THE TEACHING INTERVENTION

This section presents the findings on the nature of learners' conceptual understanding and procedural fluency in fractions before the intervention. The data presented here is drawn from the responses of learners in the pre-test, pre-interviews and recall interviews. The findings are presented in three main subsections 4.3.1, 4.3.2 and 4.3.3. For each subsection, appropriate and inappropriate responses are identified and discussed.

4.3.1 Relating unit subdivisions to the denominator of fractions

The data presented in this subsection are drawn mainly from the pre-test and partly from the pre-interview and recall interviews. Both the pre-test and pre-interview investigated three main areas of competence (see Table 4.1, Appendix B and Appendix D). This subsection presents data drawn solely from the first section of the pre-test and pre-interviews since the learners' conceptual understanding of fractions as numbers, was assessed in this section. The first section of the pre-test consisted of 11 questions. An example of a learner's responses to questions in the first section of the pre-test, are shown in Figures 4.1 and 4.2. In the pre-test and pre-interview, two major models were used to measure the conceptual understanding of fractions, namely: pre-partitioned number lines and area models (circles and rectangular fraction bars), by identifying and representing fractions less than one whole unit and fractions greater than one whole unit.

Findings from the analysis of learners' responses in the pre-test, pre-interviews and recall interviews showed that no learners (except L6 and L7) showed conceptual understanding of fractions as numbers, since they did not relate the unit subdivisions to fraction denominators. Instead, they used inappropriate fraction notation to identify fractions less than one whole unit and fractions greater than one whole unit as a count of shaded parts only. This was common in the responses of all learners in Group A and five learners in Group B. L6 and L7 in Group B were the only learners that related unit subdivisions in the models to the denominator of fractions less than one whole unit and fractions greater than one whole unit. The next two subsections discuss how the learners related the unit subdivisions to fraction denominators, using the number lines and area models.



1. Naming and Identifying fractions

1.1. Determine which shaded areas represent $\frac{1}{5}$ of the bar.

A

B

C

D

Answer: a b c

1.2. What fraction of the circle is the shaded area?

Answer: $\frac{5}{7}$

1.3. Determine which letters best describe the shaded parts of the whole circle.

A Five sevenths

B Two sevenths

C Five quarters

D Two quarters of seven

Answer: D

1.4. A fraction $\frac{3}{4}$ reads 'three quarters'.

Write the fraction $\frac{5}{12}$ in words.

Answer: Five twelfths

1.5. Below is a number line with an interval from zero to one.

What fraction is shown by the shaded length on the number line?

Answer: $\frac{1}{5}$

1.6. What fraction does the letter D point to?

Answer: $\frac{1}{2}$

1.7. Mark the position of the fraction $\frac{3}{4}$ on the number line with cross (x).

Figure 4.1: **Workings of L4 extracted from the pre-test for identifying fractions less than one whole unit**

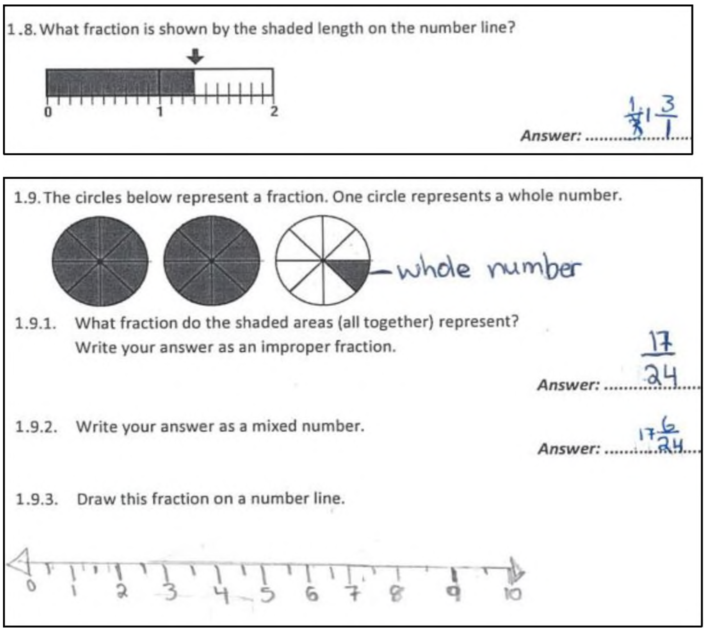


Figure 4.2: **Workings of L4 extracted from the pre-test for identifying fractions greater than one whole unit**

4.3.1.1 Inappropriate relating of unit subdivisions to fraction denominators

This subsection discusses how some learners related unit subdivisions to the denominator of fractions inappropriately, for identifying fractions less than one whole unit (including reading fractions using inappropriate names) and fractions greater than one whole unit, as well as for representing fractions greater than one unit on the number line.

Three learners in Group A (L3, L4 and L5) and two learners in Group B (L8 and L11) used inappropriate fraction notation for identifying fractions less than one unit on the number line of one unit long (see question 1.5 and 1.6 in Figure 4.1). They did not relate the number of unit subdivisions to the denominator of the fractions; instead, they used a count of shaded parts to represent the fractions in different ways. For instance, L4 identified the fractions (in questions 1.5 and 1.6) as 1/5 and 1/2 respectively, expressing the counts of shaded parts in the form *1/ (number of shaded parts)*. L3 and L11 identified the fractions shown on the number line in question 1.5 and 1.6, as 5/1 and 2/1 respectively. L5 used whole numbers to represent the shaded area (i.e. 5 and 2 respectively), which are the counts of the shaded parts. L8 used decimal notation and recorded the two fractions as 0.5 and 0.2.

In both questions (1.5 and 1.6), these learners used the number of shaded parts to identify fractions, without relating the fraction denominators to unit subdivisions. During a recall interview, I asked L4 to explain how he got *"one over five"* in question 1.5 and he said *"I only used to count like how many pieces from one and then I will get my denominator. ... Like here, I started like ... one over one, one over two, one over three, one over four and here one over five"* (Appendix H: lines 54-56 and 58-59). In a separate recall interview with L8, he explained his recording of the fractions 0.5 and 0.2, as *"I just found out that when you are counting from zero up to one, you cannot count one two three four because here is one [pointing at one on a number line]. … I thought is zero point and I did not know that it is out of eight so I counted like zero point one, zero point two, zero point three"* (Appendix H: lines 65-68).

Similarly, in the pre-test, most learners located 3/4 in question 1.7 between 3 and 4 on the pre-partitioned number line. They seemed to think of 3/4 as two independent whole numbers rather than a representation of a fractional quantity less than one whole unit on a number line. L9 located 3/4 at 1; L1, L3, L6, L7, L10, L11 and L12 located 3/4 at 3; L2 located 3/4 at 3 1/4; L4 and L5 located 3/4 at 3 3/4 while L8 located 3/4 at 4. These learners appeared to use whole numbers in some way (e.g. 3 of 3/4) for identifying unit subdivisions. For instance, during the recall interview, L8 explained that he located 3/4 at 4 because, *"I was confused sir, I just ... use four and I counted from three and then this is one two three four"* (Appendix H: lines 74-75), while L1, during the recall interview, indicated that she located 3/4 at 3 because it *"is where three is and then I thought on a number line, if the numerator is for example ... two, I can write it here [pointing at 2], because there is two"* (Appendix H: lines 87-88).

The use of whole numbers for identifying unit subdivisions was also applied during the pre-interviews. During the pre-interviews, all four learners that were interviewed – L3, L5, L6 and L7 located 2/3 at 2 on a number line with a 0-3 interval (see Appendix F: lines 27-28, 77-78, 88 and 112). In the pre-interview with L6, when I asked her to locate 2/3 on the number line, she told me that *"Actually, we were not taught about this, but then I do want to try"* (Appendix F: line 146). L6 went on and shaded the length of 2 units for locating 2/3 on the number line, and then I asked her why she located 2/3 at 2. L6 explained: *"I am saying this because in the circle, two parts are shaded and the third part is not shaded"* (Appendix F: lines 160).

Another use of inappropriate relating of unit subdivisions to the denominator of fractions was reading fractions using inappropriate names. The majority of the learners identified fractions using inappropriate names. For instance, the learners identified the fraction represented by the area shaded of a circle in question 1.3 either as *"five quarters"* or "*two quarters of seven"*. In question 1.4, these learners used three inappropriate names to describe 5/12 in words, namely: *"five over twelve"*, *"five quarters of twelve"* and *"five quarters"*. These learners used the word *"quarters"* to name fractions that were not quarters. During the recall interviews with L4 and L8, they were asked to explain how they chose their answers in question 1.3. L4 explained that he chose *"five quarters of twelve"* because *"I thought two represents the denominator and the quarter of seven represents the numerator"* (Appendix H: lines 19), while L8 explained that he chose *"five quarters"* as his answer, because *"I just forgot that a quarter is something out of four"* (Appendix H: line 33).

The inappropriate relating of unit subdivisions to the denominator of fractions using inappropriate fraction notation was also applied when identifying fractions greater than one whole unit. In the pre-test, four questions – 1.8, 1.9.1, 1.9.2 and 1.9.3, assessed learners' conceptual understanding of fractions greater than one whole unit. An example of a learner's responses to these questions is shown in Figure 4.2.

A total of nine learners (five learners from Group A and four learners from Group B) did not relate the unit subdivisions to the denominators of fractions greater than one unit. Instead, they used different, inappropriate fraction notation for identifying fractions from the area model and number line. For instance, the responses of these learners in question 1.8 of Figure 4.2, suggested that these learners did not relate the unit subdivisions to the fraction denominator when they were identifying the fraction greater than one unit on the number line. L1 recorded 1/4; L3 recorded 3 1/2; L4 recorded 1/3; L5 recorded 3/1; L8 recorded 3/10; L2 and L10 recorded 13/20; L9 recorded 1 3/2; while L11 recorded 13/2. Their fraction notation suggested that they used the count of shaded parts (either the three shaded parts after the first unit, or all thirteen shaded parts) for identifying the fraction. In a recall interview, L8 explained that to get 3/10, he *"start[ed] counting from here [pointing at 1], then I counted three for the numerator and ten parts of the denominator"* (Appendix H: lines 119-120). In the recall interview, L9 explained that she recorded 1 3/2 as her answer, because *"I did not know how to find the fraction on the number line"* (Appendix H: line 122). In the recall interview, L12 explained that he recorded 13/20, because *"I counted the shaded parts which are thirteen and for twenty, I counted all parts here"* (Appendix H: lines 79).

Similarly, the responses of these learners in question 1.9 suggested that they did not relate the unit subdivisions to the fraction denominators when they used the area model (circles) to identify fractions greater than one whole unit. For instance, in question 1.9.1, these learners identified different, inappropriate improper fractions such as 24/17 by L2 and L4; 17/24 by L5; 53/26 by L1; 17/7 by L8; 3/2 by L9; and 16/1 by L11. In a recall interview, L1 explained that she got 53/26, because "*I thought this one together is twenty six, then I add twenty six plus twenty six ... I get fifty two and then I add this one again"* (Appendix H: lines 97-98).

In question 1.9.2, these learners used inappropriate fraction notation for identifying the fraction as a mixed fraction (the same fraction as the improper fraction they identified in question 1.9.1). The analysis of their responses suggested that most learners obtained the mixed fractions by converting improper fractions (their answers in question 1.9.1) into mixed fractions. For instance, L1 recorded 2. 1/26 from 53/26; L2 and L11 both recorded 1 7/17 from 24/17 and L9 recorded 1½ from 3/2; while L3 recorded 3 17/8 by simply placing 3 before 17/8. Some learners identified inappropriate mixed fractions in question 1.9.2, but it is not clear how they obtained their answers. For instance, L10 recorded 1 1/1, although the fraction in question 1.9.1 was 16/1; L4 recorded 3 4/7, although the fraction in question 1.9.1 was 24/17; and L5 recorded 17 6/24, although the fraction in question 1.9.1 was 17/24. The responses of all nine learners suggested that these learners saw a mixed fraction as a formal symbol and lacked the conceptual representation of a mixed fraction.

Lastly, learners were asked to represent the shaded area of circles in question 1.9 as a fraction on the number line (see question 1.9.3 in Figure 4.2). Four learners in Group A (L2, L3, L4 and L5) and four learners in Group B (L1, L8, L9, and L11) did not use the unit subdivisions of the circles or the denominators of the fractions they identified in questions 1.9.1 and 1.9.2, to partition the units of the number line. Their drawings of the number line in Figure 4.2 suggested that most of these learners used the numerators of their fractions in question 1.9.1 for locating the fractions as whole numbers. In their attempt to locate the fraction, these learners plotted a different fraction from the one they had in question 1.9.1. For instance, L3 located 1 8/10 instead of 17/8; L11 located 24/10 instead of 24/17; and L4 located 24/17 between 20 and 30 on the number line; while L8 located 17/24 on the number line with unit interval of 24 unit subdivisions. In addition, the pre-interview results indicated that three out of the four learners that were interviewed – L3, L5, L6, did not relate the unit subdivisions to the denominator of fractions and they did not differentiate fractions less than one whole unit, from fractions greater than one whole unit. They identified the fractions at point A (4/6) and point B (9/6) of a number line as follows: L3 and L5 recorded A=¼ and B=4/1, while L6 recorded A=4/13 and B= 9/13 (see Appendix F: lines 73-74, 85 and 108-110).

4.3.1.2 Appropriate relating of unit subdivisions to fraction denominators

This subsection discusses how some of the learners related unit subdivisions to the denominator of fractions appropriately, for identifying fractions less than one whole unit (including reading fractions using appropriate names) and fractions greater than one whole unit, as well as for representing fractions greater than one whole unit on the number line.

The analysis of learners' responses indicated that eleven learners appropriately related the unit subdivisions to the denominator of fractions less than one whole unit, using the area model of a single unit (one circle and/or one fraction bar). For instance, in question 1.1 of Figure 4.1 all 12 learners chose 1/5 as the fraction for the area shaded of the given pre-partitioned fraction bar. In question 1.2, 11 learners identified the shaded area of a single circle correctly as 5/12, except for L4 who identified the shaded area as 5/7, which seemed to be the count of shaded parts over the count of unshaded parts, respectively. In question 1.4, only three learners – L3, L4 and L9 described 5/12 correctly as *"five twelfths"*. In questions 1.5 and 1.6, three learners in Group A (L1, L2 and L10) and four learners in Group B (L6, L7, L9 and L12) identified the shaded distance of the number line correctly as 5/8 and 2/6 respectively.

Two learners in Group B (L6 and L7), related the unit subdivisions to the denominator of fractions greater than one whole unit appropriately, using both the number line and the area model. An example of a learner's response for this is shown in Figure 4.3. In question 1.8, both learners identified the shaded length of the number line as 1 3/10 (mixed fraction). In questions 1.9.1 and 1.9.2 (see Figure 4.3), both learners correctly identified the fraction represented by the shaded area of multiple circles as an improper fraction 17/8 and as a mixed fraction 2 1/8. The responses of these learners suggested that they saw a mixed fraction as a result of converting improper fractions. The notes made by L6 in Figure 4.3 explained how

L6 converted the improper fraction 17/8 to obtain the mixed fraction 2 1/8. The responses of both learners in question 1.9.3 suggested that these learners related the unit subdivisions of the circles in question 1.9, to the denominator of fractions greater than one whole unit in questions 1.9.1 and 1.9.2 by partitioning the whole units of the number line and located 17/8 = 2 1/8 on the number line appropriately (see question 1.9.3 in Figure 4.3).

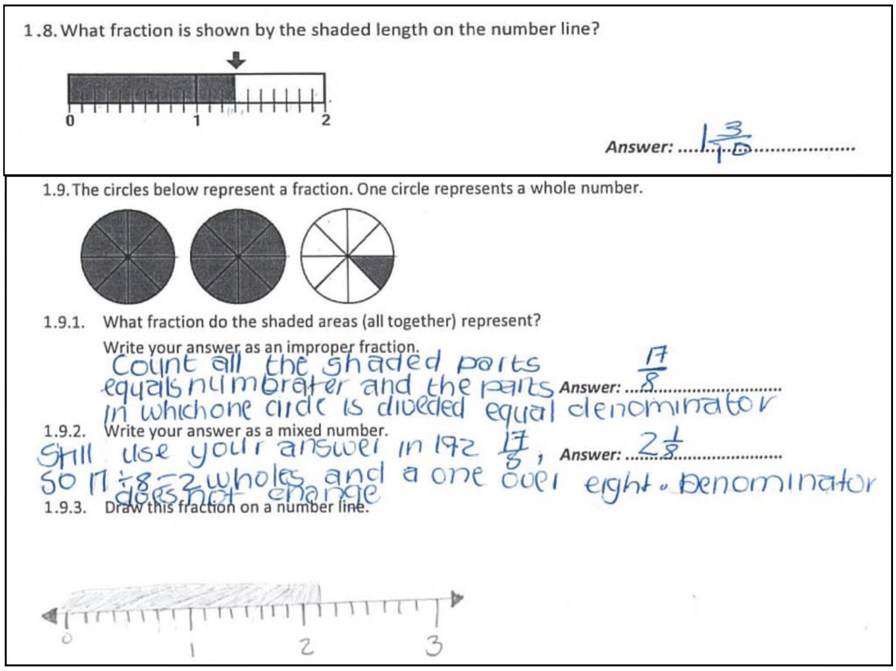
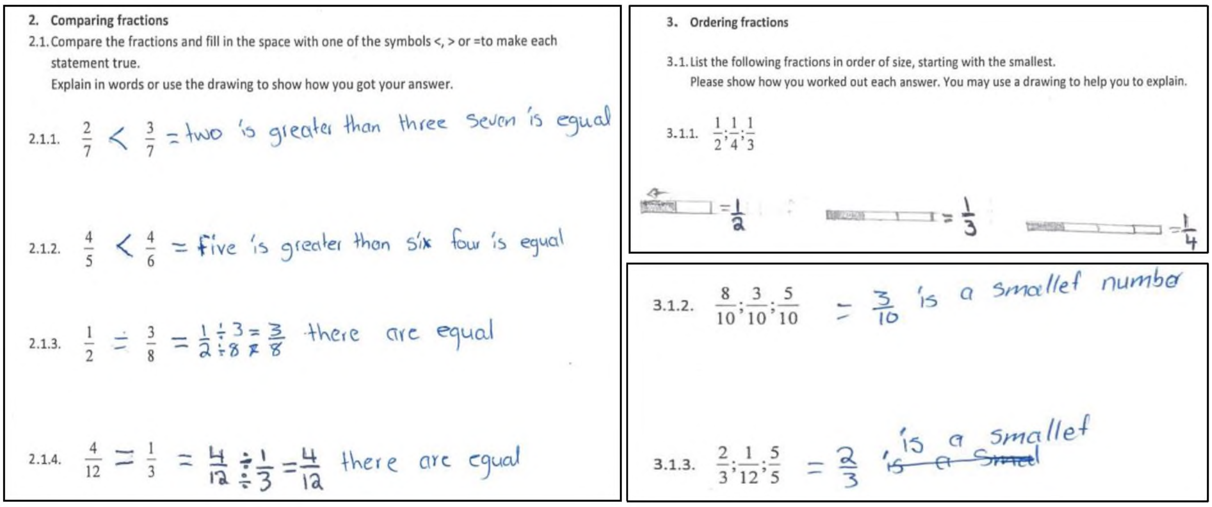


Figure 4.3: Workings of L6 extracted from the pre-test for identifying and locating fractions greater than one whole unit on the number line

4.3.2 Using procedures to compare and order fractions

This subsection presents and discusses the nature of learners' conceptual understanding of and procedural fluency with procedures they used for comparing and ordering fractions before the intervention. The use of procedures for comparing and ordering fractions was assessed in sections 2 and 3 of the pre-test, which consisted of seven questions presented in Figure 4.4. The findings from the analysis of learners' responses indicated that the majority of the learners used inappropriate procedures to compare and order fractions. The next two subsections discuss the inappropriate and appropriate procedures used to compare and order fractions.



2. Comparing fractions

2.1. Compare the fractions and fill in the space with one of the symbols <, > or =to make each statement true.
Explain in words or use the drawing to show how you got your answer.

2.1.1. $\frac{2}{7} < \frac{3}{7}$ = two is greater than three seven is equal

2.1.2. $\frac{4}{5} < \frac{4}{6}$ = five is greater than six four is equal

2.1.3. $\frac{1}{2} = \frac{3}{8} = \frac{1 \div 3}{2 \div 8} = \frac{3}{8}$ there are equal

2.1.4. $\frac{4}{12} = \frac{1}{3} = \frac{4 \div 1}{12 \div 3} = \frac{4}{12}$ there are equal

3. Ordering fractions

3.1. List the following fractions in order of size, starting with the smallest.
Please show how you worked out each answer. You may use a drawing to help you to explain.

3.1.1. $\frac{1}{2};\frac{1}{4};\frac{1}{3}$

$=\frac{1}{2}$ $=\frac{1}{3}$ $=\frac{1}{4}$

3.1.2. $\frac{8}{10};\frac{3}{10};\frac{5}{10}$ $= \frac{3}{10}$ is a smallef number

3.1.3. $\frac{2}{3};\frac{1}{12};\frac{5}{5}$ $= \frac{2}{3}$ is a smallef ~~is a Small~~

Figure 4.4: Workings of L5 extracted from the pre-test for comparing and ordering fractions

4.3.2.1 Inappropriate procedures used to compare and order fractions

The analysis of learners' responses in the pre-test and pre-interview identified three inappropriate procedures that were used to compare and order fractions. In Table 4.2, the distribution of these inappropriate procedures is highlighted in grey. The first inappropriate procedure involved using the sizes of denominators only. In Table 4.2, this procedure is represented by two keys, namely: *Used D* and *Big D small* (the description of these keys is given in Table 4.2). The second inappropriate procedure involved using the numerators as divisors and the denominators as dividends. In Table 4.1, this procedure is represented by the key: *Small N big*. The third inappropriate procedure involved using inappropriate arithmetic manipulation (IARM) to equate the given fractions, though the fractions are not really equal. In Table 4.2, this procedure is represented by the key: *IARM* (find the description of the key in Table 4.2).

Table 4.2: Distribution of inappropriate procedures used for comparing and ordering fractions in the pre-test

The frequency of inappropriate procedures used for comparing and ordering fractions													
Procedural skills tested	Types of fractions per question in the pre-test	Responses of learners											
		L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Comparing two fractions	2.1.1 Fractions with same denomintor	√	Small N big	√	√	√	√	√	Small N big	Small N big	√	IARM	√
	2.1.2 Fractions with same numerator	Used D	Small N big	√	Big D small	Used D	√	√	√	√	√	Used D	√
	2.1.3 Fractions with different numerators and denominators	Big D small	Small N big	√	Big D small	IARM	√	√	√	√	√	Used D	√
	2.1.4 Equivalent fractions	Big D small	Small N big	√	Big D small	IARM	√	√	√	√	√	Used D	√
Ordering three fractions	3.1.1 Fractions with same numerator	√	√	√	√	Used D	√	Used D	√	√	√	Used D	√
	3.1.2 Fractions with same denominator	√	Small N big	Small N big	√	√	√	√	√	Small N big	√	√	√
	3.1.3 Fractions with different numerators and denominators	√	Small N big	Small N big	Big D small	Used D	√	√	No answer	Small N big	√	√	√
Keys													
√	The learner compared fractions using an appropriate procedure												
Used D	The learner compared fractions using the sizes of denominators only such that the *bigger fraction* is the one with the *bigger denominator*												
Big D small	The learner compared fractions using the sizes of denominators only such that *the smaller fraction* is the one with *the bigger denominator*												
Small N big	The learner compared fractions using *numerators as divisors* and *denominators as dividends* such that the *bigger fraction* is the one with a *smaller numerator*												
IARM	The learner compared fractions using *inappropriate arithmetic manipulation* to equate the given fractions, although they are not really equal												
Hint: Grey shading indicates the use of inappropriate procedures for comparing and ordering fractions													

4.3.2.1.1 Using the sizes of denominators only

This procedure was applied in two ways. The first method involved using denominators so that the bigger fraction is the one with the bigger denominator and the smaller fraction is the one with the smaller denominator. This procedure was used by four learners (L1 and L5 in Group A, and L7 and L11 in Group B) to compare and order fractions with the same numerators (see the distribution of the key *Used D* in Table 4.2). In the pre-test, these learners concluded that 4/5<4/6 and ½<1/3<1/4 in questions 2.1.2 and 3.1.1 respectively (see Figure 4.4). In the recall interview, L1 explained that he obtained 4/5<4/6, because *"I thought four sixths is bigger, because the denominator is bigger"* (Appendix H: line 111). This method was also applied to equivalent fractions and fractions with different numerators and denominators by L5 and L11. L11 concluded that ½<3/8 and 4/12>1/3 (see questions 2.1.3 and 2.1.4 of Figure 4.4), while L5 concluded that 2/3<5/5<1/12(see question 3.1.3 of figure 4.4).

The second method involved using the sizes of denominators so that the smaller fraction is the one with a bigger denominator and the bigger fraction is the one with a smaller denominator (find the distribution of the key *Big D small* in Table 4.2). This method was used by only two learners (L1 and L4) from Group A. For instance, in questions 2.1.4 and 3.1.3 of Figure 4.4, L1 and L4 concluded that 4/12<1/3 (these are equivalent fractions) and 1/12<5/5<2/3 respectively.

4.3.2.1.2 Using the numerators as divisors and the denominators as dividends

This procedure was used by four learners (L2, L3 and L8 from Group A, and L9 from Group B). These learners used numerators as divisors and denominators as dividends for comparing and ordering fractions. In Table 4.2, this procedure is represented by the key: *Small N big.* For instance, in the pre-test, in questions 2.1.1, 2.1.2, 2.1.3, 3.1.2 and 3.1.3 of Figure 4.4, these learners concluded that 2/7>3/7; 4/5<4/6; 4/12<1/3; 8/10<5/10<3/10; and that 5/5<2/3<1/12 respectively. In the recall interview, L2 explained that he thought 2/7>3/7, because *"first, I thought that the fractions with smallest numerators are always the bigger ones"* (Appendix H: lines 171). In the recall interview, L9 also explained that he thought that 2/7>3/7 and 8/10<5/10<3/10, because *"I thought when the denominators are the same, the smaller the numerator the bigger the fraction. ... I thought... ten people who are going to divide in three parts [three tenths] are the one who are going to get the biggest, and this one [eight tenths] is the smallest one"* (Appendix H: lines 196-198 and 202-205). Figure 4.5 presents the workings of L9 and L2 for applying this procedure to compare and order fractions inappropriately.

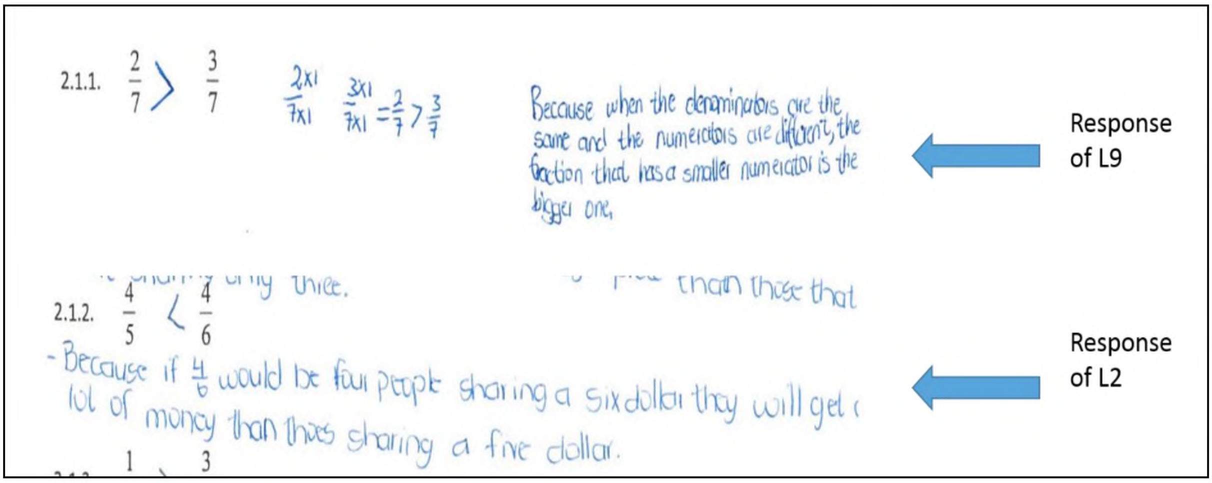


Figure 4.5: Workings of L9 and L2 extracted from the pre-test for using the numerators as divisors and denominators as dividends

4.3.2.1.3 Using inappropriate arithmetic manipulation

This procedure involved using inappropriate arithmetic manipulation of the given fractions to equate the fractions, although they are not equal. In Table 4.2, this procedure was used by L5 and L11 for comparing and ordering fractions. For instance, in the pre-test, these learners concluded that 2/7=3/7, ½=3/8 and 4/12=1/3. A good illustration of how this procedure was applied is presented in Figure 4.4 showing the workings of L5.

4.3.2.2 Appropriate procedures used to compare and order fractions

This subsection presents and discusses four appropriate procedures used for comparing and ordering fractions as identified from the analysis of learners' responses in the pre-test and pre-interview. In Table 4.3, these procedures are represented by one of the following keys: *LCD, UF.bars or U.circles, Rules* and *Sharing*. Here, only three procedures (*LCD, UF.bars or U.circles and Rules)* are discussed in detail (in subsections 4.3.2.2.1 - 4.3.2.2.3), because these three procedures were frequently used. The fourth procedure represented by the key "*Sharing*" was used only twice by L4 (see Table 4.3), and thus, it is discussed briefly at the end of subsection 4.3.2.2.3.

Table 4.3: Distribution of appropriate procedures used for comparing and ordering fractions in the pre-test

The frequency of appropriate procedures used for comparing and ordering fractions													
Procedural skill tested	Types of fractions per question in the pre-test	Responses of learners											
		L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Comparing two fractions	2.1.1 Fractions with same denomintor	Answer only	X	LCD	Sharing	Rules	Rules	Rules	X	X	Rules	X	UF.bars
	2.1.2 Fractions with same numerator	X	X	LCD	X	X	LCD	Rules	LCD	LCD	Rules	X	UF.bars
	2.1.3 Fractions with different numerators and denominators	X	X	LCD	X	X	LCD	LCD	LCD	LCD	UF.bars	X	UF.bars
	2.1.4 Equivalent fractions	X	X	LCD	X	X	LCD	LCD	LCD	LCD	UF.bars	X	UF.bars
Ordering three fractions	3.1.1 Fractions with same numerator	Answer only	Answer only	LCD	Sharing	X	U.circles	X	LCD	LCD	UF.bars	X	UF.bars
	3.1.2 Fractions with same denominator	Answer only	X	X	Answer only	Rules	UF.bars	UF.bars	LCD	X	UF.bars	UF.bars	UF.bars
	3.1.3 Fractions with different numerators and denominators	Answer only	X	X	X	X	UF.bars	LCD	X	X	UF.bars	UF.bars	UF.bars

Keys	
LCD	The learner compared fractions by converting the given fractions to fractions with a *(lowest) common denominator*
UF.bars or U.circles	The learner compared fractions using unequal *fraction bars* (UF.bars) *or unequal circles* (U.circles) to determine the sizes of the given fractions
Rules	The learner compared fractions using *rules* for comparing fractions with same numerator and fractions with same denominator
Sharing	The learner compared fractions using the idea of *sharing* one object between many people
Answer only	The learner compared fractions appropriately by giving the *answer only* without showing workings for the answer
Hint: Grey shading with a cross X indicates the use of inappropriate procedures for comparing and ordering fractions	

4.3.2.2.1 Converting the given fractions to fractions with a (lowest) common denominator

This procedure involved converting the given fractions to fractions with a common denominator to determine the bigger fraction so that the bigger fraction is the one with the bigger numerator for all fractions with a (lowest) common denominator *(LCD)*. Table 4.3 shows that this procedure was applied by five learners, namely: L3 and L8 from Group A and L6, L7 and L9 from Group B. Most learners obtained the common denominator by cross multiplying the denominators of the given fractions. Thereafter, learners would conclude that a fraction with a bigger numerator is greater than a fraction with a smaller numerator. Figure 4.6 shows the workings of L3 using this procedure to compare fractions in the pre-test. In questions 2.1.2 - 2.1.4 and 3.1.1 - 3.1.3, these learners concluded that 4/5>4/6, because 24/30>20/30 and 24>20; that 1/2>3/8; 4/12=1/3; 1/4<1/3<1/2; 3/10<5/10<8/10; and 1/12<2/3<5/5.

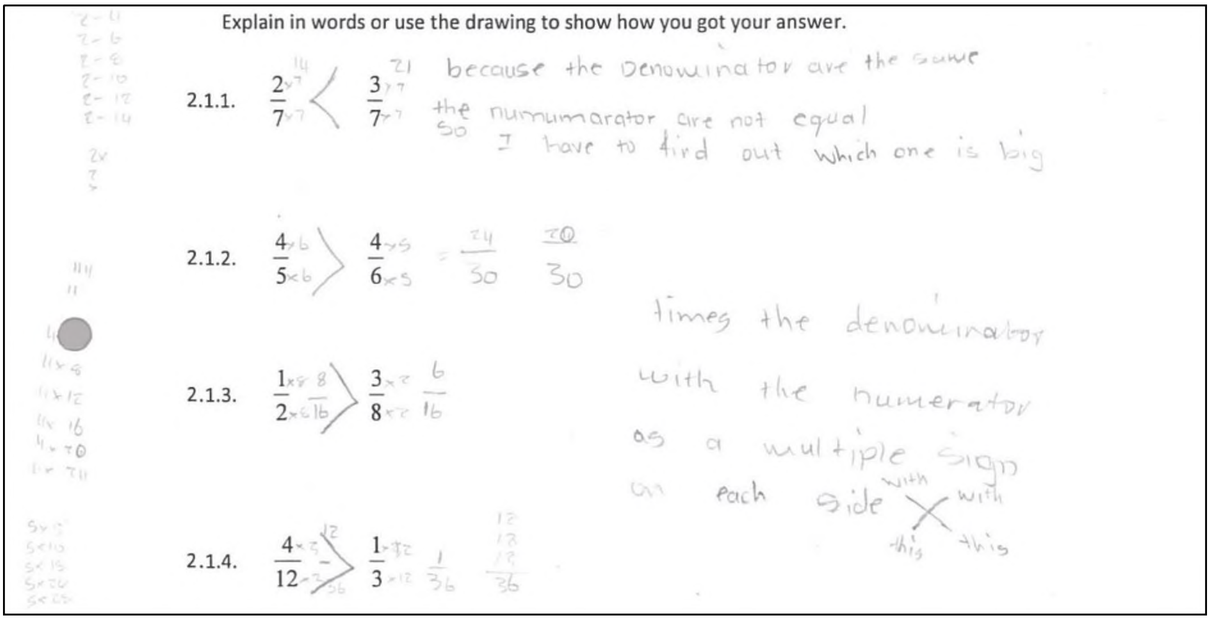


Figure 4.6: Workings of L3 extracted from the pre-test for comparing fractions by converting the given fractions to fractions with a common denominator

4.3.2.2.2 Using unequal fraction bars to determine the sizes of the given fractions

This procedure involved using unequal fraction bars and circles to determine the sizes of the given fractions, so that the bigger fraction is the one represented by the bigger shaded area of the fraction bar. In Table 4.3, this procedure is represented by the keys: *UF.bars* and *U.circles*. In the pre-test, this procedure was used by five learners, namely: L10 from Group

A and L6, L7, L11 and L12 from Group B (see the distribution of the keys *UF.bars* and *U.circles* in Table 4.3). These learners drew fraction bars or circles, which they partitioned according to the denominator of the respective fractions, and thereafter shaded the appropriate number of unit subdivisions (equivalent to the numerator of the given fractions). Based on the proportion of the shaded area, these learners would determine the appropriate bigger fraction. This procedure was used to compare and order all types of fractions listed in Table 4.3, namely: fractions with same denominator; fractions with same numerator; equivalent fractions; and fractions with different numerators and denominators. An example of a learner's responses using this procedure is shown in Figure 4.7.

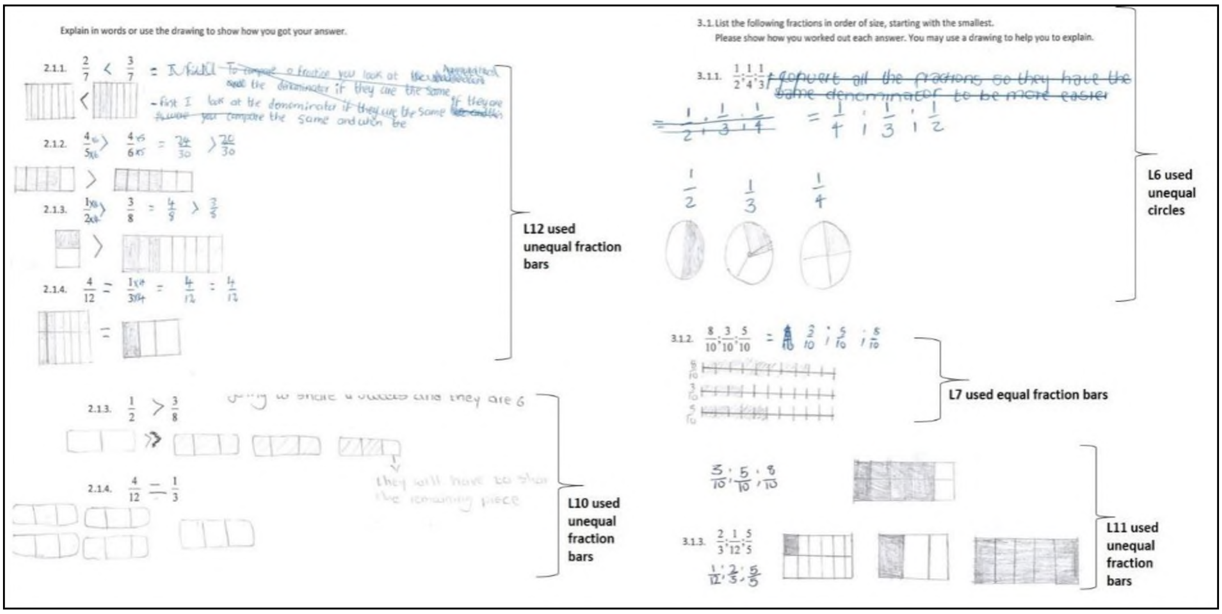


Figure 4.7: Workings of L12, L10, L6, L7 and L11 extracted from the pre-test for comparing and ordering fractions using unequal fraction bars and unequal circles

4.3.2.2.3 Using rules for comparing fractions with the same numerator or denominator

This procedure involved using two rules. The first rule is for comparing fractions with the same numerator so that "when the fractions have the same numerator, the bigger fraction is the one with the smaller denominator, while the smaller fraction is the one with a bigger denominator." Table 4.3 shows that this rule was used in the pre-test by two learners, namely: L10 from Group A and L7 from Group B. In question 2.1.2 of the pre-test, these learners used this rule to conclude that 4/5>4/6. An example showing the workings of a learner's response (question 2.1.2) is shown in Figure 4.8.

The second rule is for comparing and ordering fractions with the same denominator such that "when the fractions have the same denominator, the bigger fraction is the one with the bigger numerator, while the smaller fraction is the one with a smaller numerator." Table 4.3 shows that this rule was used in the pre-test by four learners, namely: L5 and L10 from Group A and, L6 and L7 from Group B. In question 2.1.1 of the pre-test, these learners used this rule to conclude that 2/7<3/7. In question 3.1.2 of the pre-test, L5 used this rule to conclude that 3/10<5/10<8/10. An example showing the workings of a learner's response (question 2.1.1) is shown in Figure 4.8.

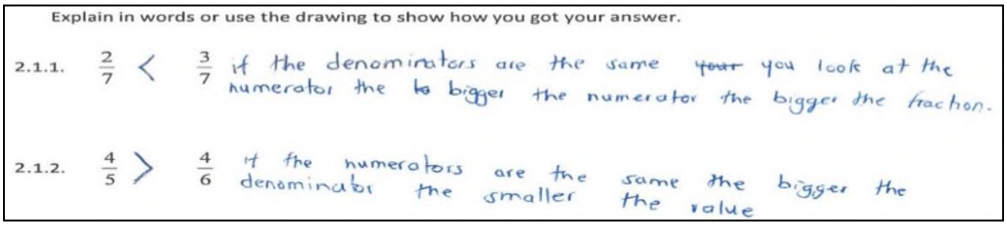


Figure 4.8: The snapshot of how L7 applied rules for comparing fractions with same numerator and fractions with same denominator in the pre-test

The fourth appropriate procedure involved using the idea (knowledge) of sharing one object between many people (e.g. 1/4 represents four people sharing one object (e.g. bread) and 1/3 represents three people sharing one object (e.g. same sized bread as for four people); then three people (1/3) would get bigger pieces than four people (1/4) and therefore, 1/4<1/3). In the pre-test, this procedure was used by L4 only. In questions 2.1.1 and 3.1.1 of the pre-test, L4 used this procedure to conclude that 2/7<3/7 and 1/4<1/3<1/2 respectively. An example showing the workings of L4 in the pre-test, for comparing fractions using this procedure is shown in Figure 4.9.

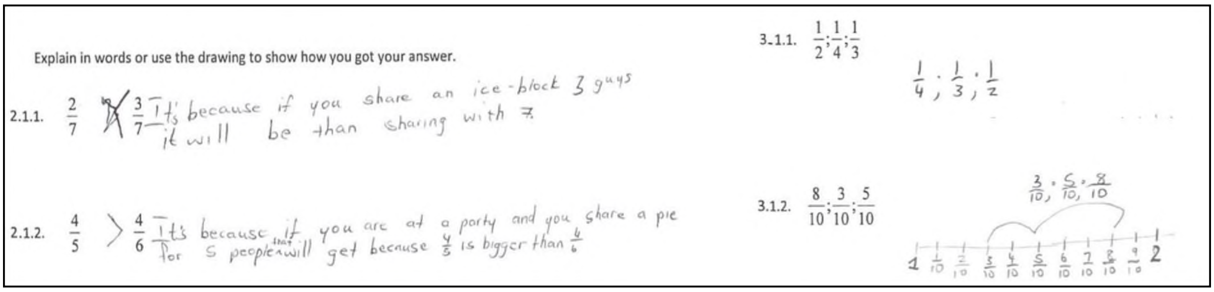


Figure 4.9: Workings of L4 extracted from the pre-test for comparing fractions using "sharing"

4.3.3 Adding two fractions

This subsection presents and discusses the nature of learners' conceptual understanding of and procedural fluency with the procedures they used for adding two fractions before the intervention. The use of procedures for adding two fractions was assessed in section 4 of the pre-test, which consisted of two questions, presented in Figure 4.10. The analysis of learners' responses in the pre-test and pre-interview indicated that majority of the learners in Group A used inappropriate procedures for adding two fractions. The next two subsections present and discuss the inappropriate and appropriate procedures used for adding two fractions with the same denominator and fractions with different denominators.

4.3.3.1 Inappropriate procedures used for adding two fractions

This subsection presents and discusses three inappropriate procedures used for adding two fractions, as identified from the analysis of learners' responses in the pre-test and pre-interview. In table 4.4, these inappropriate procedures are highlighted in grey and represented by one of the following keys: *ANDT, UN and IARM.* Subsections 4.3.3.1.1 - 4.3.3.1.3 discuss in detail the three inappropriate procedures used for adding fractions in the pre-test and pre-interview.

Table 4.4: Distribution of inappropriate procedures used for adding two fractions in the pre-test

The frequency of inappropriate procedures used for adding two fractions													
Procedural skills tested	Types of fractions per question in the pre-test	Responses of learners											
		L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Adding two fractions	4.1.1 Fractions with same denominator	√	√	√	√	ANDT	√	√	√	√	IARM	√	√
	4.1.2 Fractions with different numerators and denominators	√	IARM	UN	√	ANDT	√	√	√	UN	IARM	UN	√

Keys	
√	The learner added fractions appropriately using the appropriate procedure
ANDT	The learner added fractions inappropriately by adding numerators together and adding denominators together
UN	The learner added fractions by adding unchanged numerators (UN) together after obtaining the appropriate common denominator
IARM	The learner added fractions using inappropriate arithmetic manipulation (IARM)
Hint: Grey shading indicates the use of inappropriate procedures for adding fractions	

4.3.3.1.1 Adding numerators together and denominators together

This procedure involved finding the sum of two fractions by adding numerators together and denominators together. This procedure is represented in Table 4.4 by the key: *ANDT*. Table 4.4 shows that this procedure was used only by L5 from Group A. Figure 4.10 illustrates the workings of L5 using this procedure to add fractions with same denominator and fractions with different denominators. In the pre-interview, L5 applied the procedure to conclude that 2/5+1/6=3/12 (see Appendix F: lines 231-232).

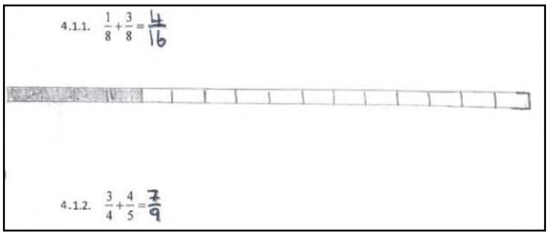
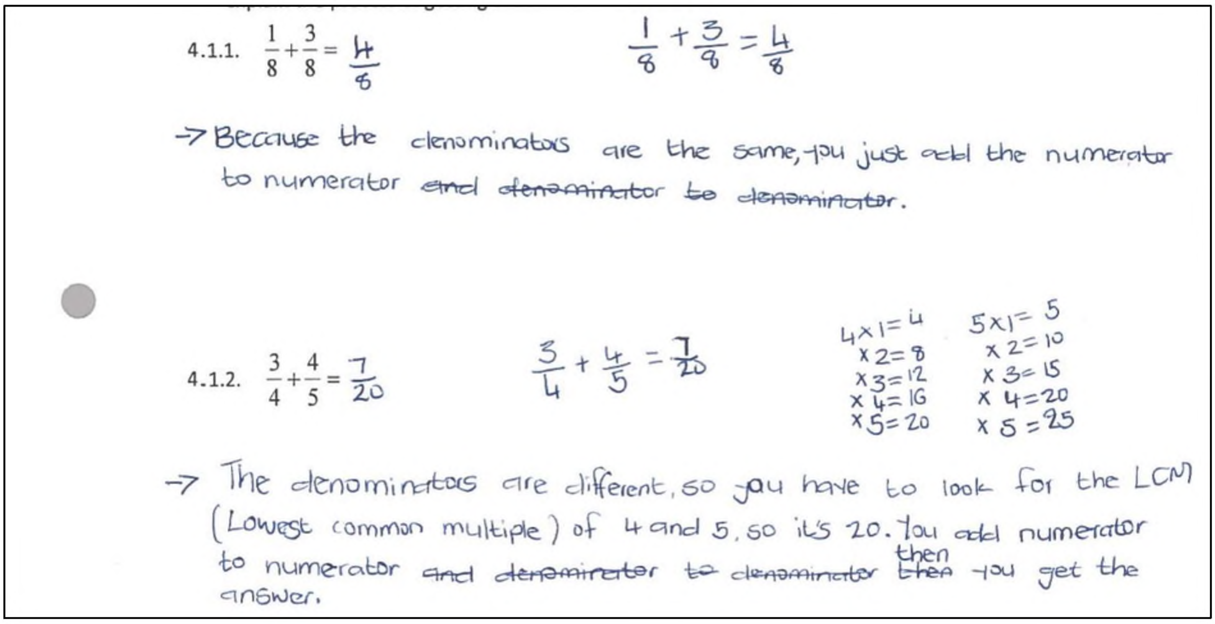


Figure 4.10: Workings of L5 extracted from the pre-test for adding two fractions inappropriately by adding numerators together and adding denominators together

4.3.3.1.2 Adding fractions with a new common denominator and unchanged numerators

This incorrect procedure involved finding the sum of two fractions with different denominators by adding fractions with a new common denominator and unchanged numerators. This procedure is represented in Table 4.4 by the key: *UN*. Table 4.4 shows that this procedure was applied by three learners, namely: L3 from Group A and, L9 and L11 from Group B. An example showing the workings of a learner using this procedure in the pre-test, is shown in Figure 4.11. The workings shown in Figure 4.11 suggest that these learners obtained the sum of fractions by adding unchanged numerators together once they calculated the lowest common denominator.


4.1.1. $\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$ $\frac{1}{8}+\frac{3}{9}=\frac{4}{8}$

-> Because the denominators are the same, you just add the numerator to numerator ~~and denominator to denominator~~.

4.1.2. $\frac{3}{4}+\frac{4}{5}=\frac{7}{20}$ $\frac{3}{4}+\frac{4}{5}=\frac{7}{20}$

$4\times1=4$, $\times2=8$, $\times3=12$, $\times4=16$, $\times5=20$

$5\times1=5$, $\times2=10$, $\times3=15$, $\times4=20$, $\times5=25$

-> The denominators are different, so you have to look for the LCM (Lowest common multiple) of 4 and 5, so it's 20. You add numerator to numerator ~~and denominator to denominator~~ then ~~then~~ you get the answer.

Figure 4.11: Workings of L11 extracted from the pre-test for finding the sum of two fractions by adding fractions with a new common denominator and unchanged numerators

4.3.3.1.3 Using inappropriate arithmetic manipulation

This procedure for adding fractions involved using inappropriate arithmetic manipulation. This procedure is represented in Table 4.4 by the key: *IARM*. Table 4.4 shows that this procedure was used by two learners from Group A, namely: L2 and L10. An example showing the workings of a learner using this procedure in the pre-test is shown in Figure 4.12. The responses shown in Figure 4.12 were difficult to interpret, and thus, these responses were described as inappropriate arithmetic manipulation.

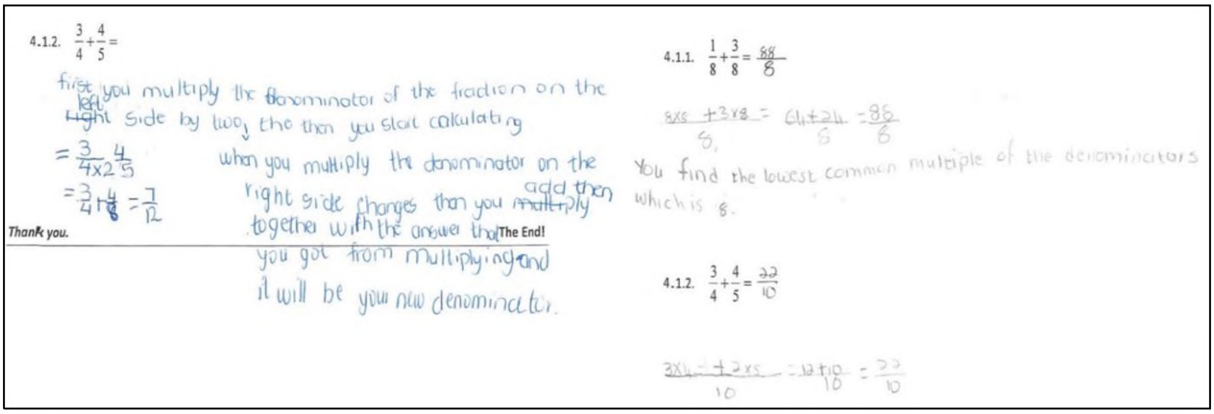


Figure 4.2: Workings of L2 and L10 (from left to the right) extracted from the pre-test showing the use of inappropriate arithmetic manipulation for adding fractions

4.3.3.2 Appropriate procedures of adding two fractions

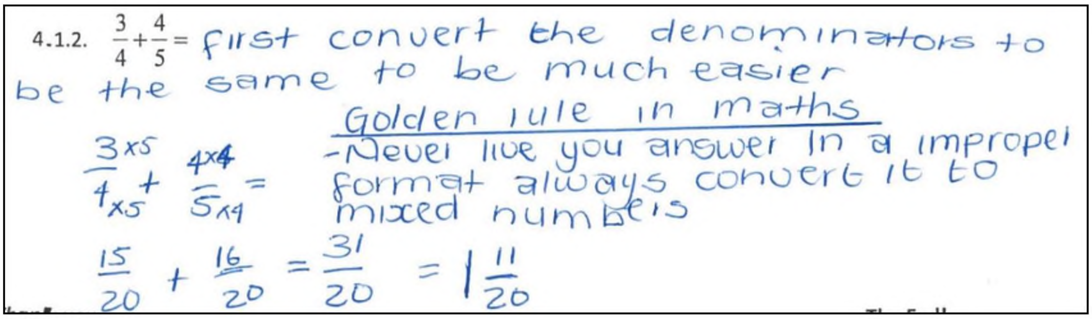
This subsection presents and discusses two appropriate procedures used for adding fractions, as identified from the analysis of learners' responses in the pre-test and pre-interview. These procedures are represented in Table 4.5 by one of the two keys: *PF* and *CU*. Subsections 4.3.3.2.1 and 4.3.3.2.2 present and discuss in detail the two procedures used for adding fractions appropriately.

Table 4.5: Distribution of the appropriate procedures used for adding two fractions in the pre-test

The frequency of appropriate procedures used for adding two fractions													
Procedural skills tested	Types of fractions per question in the pre-test	Responses of learners											
		L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Adding two fractions	4.1.1 Fractions with same denominator	PF	PF	PF	PF	X	PF	PF	PF	PF	X	PF	CU
	4.1.2 Fractions with different numerators and denominators	PF	X	X	PF	X	PF	PF	PF	X	X	X	CU
Keys													
X	The learner added fractions using inappropriate procedures												
PF	The learner displayed appropriate *procedural fluency* for adding fractions appropriately using the lowest common denominator method												
CU	The learner displayed appropriate *conceptual understanding* and procedural fluency for adding fractions by visually representing the lowest common method for adding fractions using equal fraction bars												
Hint: Grey shading indicates the use of inappropriate procedures for adding fractions													

4.3.3.2.1 Using the lowest common denominator method for adding fractions

This procedure involved using the lowest common denominator method for adding fractions with same denominator and fractions with different denominators. This procedure is represented in Table 4.5 by the key: *PF*. Table 4.5 shows that this procedure was used by nine learners, namely: three learners (L2, L3 and L4) from Group A and six learners (L1, L6, L7, L8, L9 and L11) from Group B. An example showing the response of a learner using this procedure in the pre-test is shown in Figure 4.13.


4.1.2. $\frac{3}{4}+\frac{4}{5}=$ first convert the denominators to be the same to be much easier

Golden rule in maths

-Never live you answer in a improper format always convert it to mixed numbers

$\frac{3\times5}{4\times5}+\frac{4\times4}{5\times4}=$

$\frac{15}{20}+\frac{16}{20}=\frac{31}{20}=1\frac{11}{20}$

Figure 4.3: Workings of L11 extracted from the pre-test showing the appropriate use of the lowest common denominator method for adding fractions

4.3.3.2.2 Using fraction bars to visually represent the lowest common denominator method for adding fractions

This procedure involved using equal fraction bars to visually represent the lowest common denominator method for adding fractions with same denominator and fractions with different denominators. This procedure is represented in Table 4.5 by the key: *CU*. Table 4.5 shows that this procedure was used by L12 from Group B. Figure 4.14 shows that L12 used equal fraction bars to show that 1/8+3/8=4/8 and 3/4+4/5=31/20=1 11/20. The appropriate use of fraction bars to represent the lowest common denominator method for adding fractions indicates that L12 seemed to understand that only equally sized units are countable and finding the lowest common denominator when adding fractions with different denominators is equivalent to finding the total number of equally sized subdivisions in the unit.

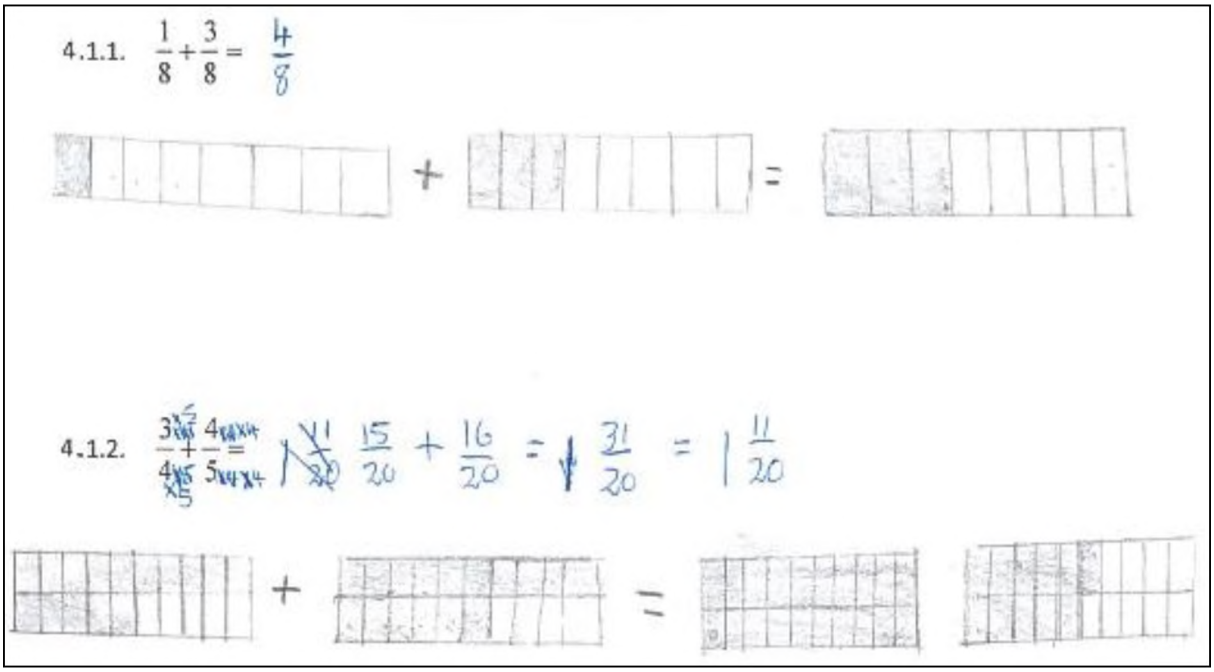


Figure 4.4: Workings of L12 extracted from the pre-test showing the appropriate use of the fraction bars to represent the lowest common denominator method for adding fractions

4.4 CHANGES IN LEARNERS AFTER THE TEACHING INTERVENTION

This section presents the analysis of the changes in learners' conceptual understanding of and procedural fluency with fractions over the period of the intervention. The data presented here are drawn from the responses of learners in the post-test, post-interviews and recall interviews. The analysis is presented in three main subsections: 4.4.1, 4.4.2 and 4.4.3. For each subsection, appropriate and inappropriate responses are identified and discussed.

4.4.1 Relating unit subdivisions to the denominator of fractions

This subsection presents and discusses the development in learners' conceptual understanding of fractions during the period of the intervention. This conceptual understanding of fractions involved using pre-partitioned number lines and area models for identifying and representing fractions less than one whole unit and fractions greater than one whole unit. It was assessed in the first section of the post-test and post-interview. Here, the discussion follows the sequence of questions in the first section of the post-test, while data drawn from the post-interviews and recall interviews are used to substantiate data from the post-test.

4.4.1.2 Appropriate relating of unit subdivisions to fraction denominators

This subsection discusses how the learners appropriately related unit subdivisions to the denominator of fractions, to identify and represent fractions less than one whole unit (including reading fractions using appropriate names) and fractions greater than one whole unit on the number lines and area models.

The appropriate relating of unit subdivisions to the denominator of fractions less than one whole unit and fractions greater than one whole unit is represented in Table 4.6 by the key: √. Table 4.6 shows that the appropriate relating of unit subdivisions to the denominator of fractions was attained by 10 learners, namely: three learners (L2, L3 and L10) from Group A and all learners (L1, L6, L7, L8, L9, L11 and L12) from Group B. An example of a learner's responses that appropriately related unit subdivisions to the denominator of fractions is shown in Figure 4.15.

Table 4.6: Distribution of appropriate responses for relating unit subdivisions to the denominator of fractions in the post-test

The frequency of appropriate responses for relating unit subdivisions to the denominator of fractions													
Conceptual understanding tested	**Types of fractions per question in the post-test**	**Responses of learners**											
		L1	**L2**	**L3**	**L4**	**L5**	**L6**	**L7**	**L8**	**L9**	**L10**	**L11**	**L12**
Fractions less than one whole unit	1.1 Writing the fraction in words using the cirlce's area shaded	√	√	√	√	√	√	√	√	√	√	√	√
	1.2 Writing the fraction in words using the length shaded length of the number line	√	√	√	√	√	√	√	√	√	√	√	√
	1.3 Identifying a proper fraction from the number line	√	√	√	√	√	√	√	√	√	√	√	√
	1.4 Locating a proper fraction on a number line	√	√	√	√	√	√	√	√	√	√	√	√
Fractions greater than one whole unit	1.5 Identifying an improper fraction from a number line	√	√	√	√	2/7	√	√	√	√	√	√	√
	1.6.1 Identifying an improper fraction from circles	√	√	√	21/17	3/21	√	√	√	√	√	√	√
	1.6.2 Identifying a mixed fraction from circles	√	√	√	2 2/7	√	√	√	√	√	√	√	√
	1.6.3 Locating the fraction in questions 1.6.1 and 1.6.2 on a number line	√	√	√	Number line not clear	√	√	√	√	√	√	√	√
Keys													
√	The learner identified or represented the fractions appropriately												
Number line not clear	The learner drew a number line which is difficult to interpret												
Hint: Grey shading shows the inappropriate relating of unit subdivisions to the denominator of fractions													

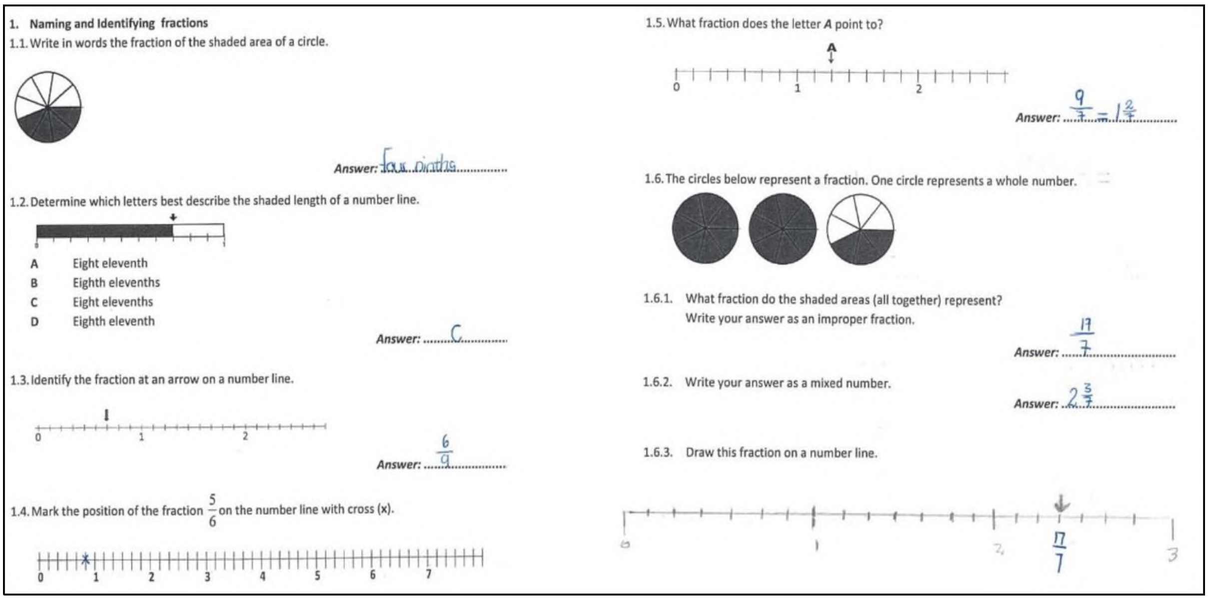


Figure 4.15: Workings of L9 extracted from the post-test showing appropriate relating of unit subdivisions to the denominator of fractions

These learners related unit subdivisions to the denominator of fractions less than one whole unit and fractions greater than one whole unit, by reading fractions using appropriate fraction names. For instance, in Figure 4.15, these learners described the fractions in questions 1.1 and 1.2 using appropriate names such as *"four ninths"* and *"eight elevenths"* respectively. In the post-interview, all four learners that were interviewed, identified and described the fraction 7/8 as "*seven eighths"* (see Appendix G: lines 3-5 and 9).

These learners also related unit subdivisions to the denominator of fractions less than one whole unit and fractions greater than one whole unit, by using appropriate fraction notation and locating fractions appropriately on the number lines. For instance, in Figure 4.15, they identified the fractions in questions 1.3 and 1.5 as 6/9 and 9/7 = 1 2/7 respectively, using the number line. These learners located 5/6 and 17/7 = 2 3/7 on a pre-partitioned number line and self-drawn number line in questions 1.4 and 1.6.3 respectively. The locating of fractions less than one unit on a number line shows a sizable change during the intervention period, since no single learner located a fraction less than one whole unit in the pre-test (see paragraph 4 in subsection 4.3.1.1). In the recall interviews, all 12 learners indicated that the drawing of the number line for locating fractions was the most salient thing they learnt from the intervention (see Appendix H: lines 30-31 and 37-40). For instance, L5 explained that he learned *"defining fractions and drawing a number line"* (Appendix H: line 23). L4 explained that he learned how to find *"or... locate a fraction on a number line. ... Now I know like on a number line how many lines should be on a number line to give you the denominator... and for a fraction like two tenths, I need to count two times, one for the numerator and the other for the denominator"* (Appendix H: lines 50 and 13-14). L7 explained that he learned *"using the denominator of a fraction to work out the partitioning of a whole"* (Appendix H: line 62). L6 indicated that she learned that *"when you are going to draw a number line ... all proper fractions are before one and then between zero and one"* (Appendix H: lines 110-115).

Finally, the responses of these learners for questions 1.6.1, 1.6.2 and 1.6.3 in Figure 4.15 suggested that these learners showed a developing conceptual understanding of improper fractions and mixed fractions as representations for expressing fractional quantities bigger than one whole unit. For instance, in the recall interview, L11 explained that she *"learned that improper fractions are always written after one [on the number line]"* (Appendix H: line 128).

4.4.1.3 Inappropriate relating of unit subdivisions to fraction denominators

This subsection discusses the inappropriate relating of unit subdivisions to the denominator of fractions. The analysis of the post-test and post-interviews show that this error only occurred when identifying and representing fractions greater than one whole unit on the number lines and area models. This inappropriate relating of unit subdivisions to the denominator of fractions is highlighted in grey in Table 4.6.

Table 4.6 shows that the inappropriate relating of unit subdivisions to the denominator of fractions greater than one whole unit occurred for two learners (L4 and L5) from Group A. An example of a learner's responses for inappropriate relating of unit subdivisions to the denominator of fractions is shown in Figure 4.16. These learners used inappropriate fraction notation when identifying fractions using number lines and circles. For instance, in questions 1.5 and 1.6.1 of figure 4.16, L5 identified the improper fractions as 2/7 and 3/21 respectively. L4 identified the fractions in questions 1.6.1 and 1.6.2 of figure 4.16 as 21/17 and 2 2/7 respectively. Both learners identified the fraction in question 1.6.2 with a denominator different from the denominator of the fraction they identified in question 1.6.1. This difference suggested that these learners did not recognise that both the mixed fraction notation and improper fraction notation are two ways of expressing the same quantity – which was bigger than one whole unit.

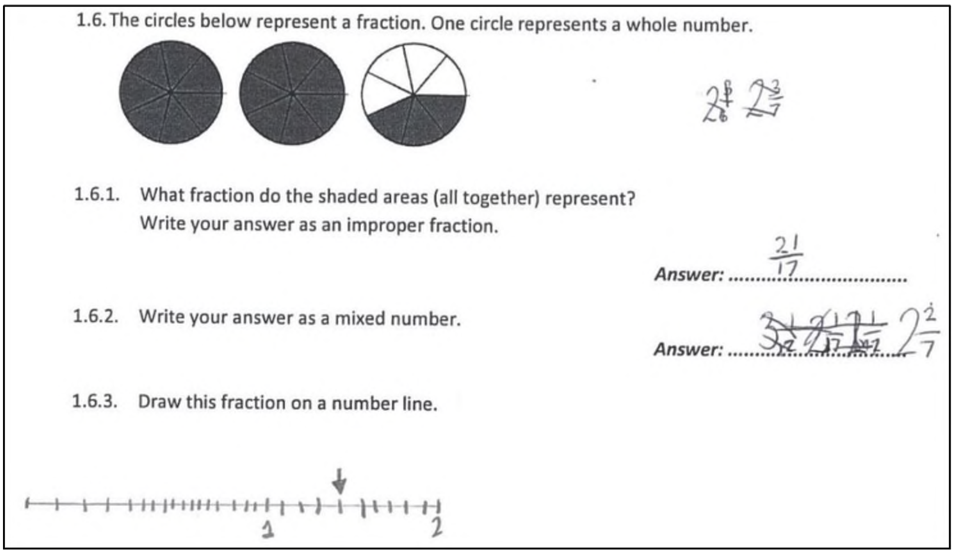


Figure 4.16: Workings of L4 extracted from the post-test showing inappropriate relating of unit subdivisions to the denominator of fractions greater than one whole unit

4.4.2 Using procedures to compare and order fractions

This subsection presents and discusses the nature of learners' conceptual understanding of and procedural fluency with the procedures they used to compare and order fractions, after the intervention. These procedures were assessed in sections 2 and 3 of the post-test, which consisted of five questions presented in Figure 4.17.The next two subsections: 4.4.2.1 and 4.4.2.2 present and discuss the appropriate and inappropriate procedures used to compare and order fractions after the intervention.

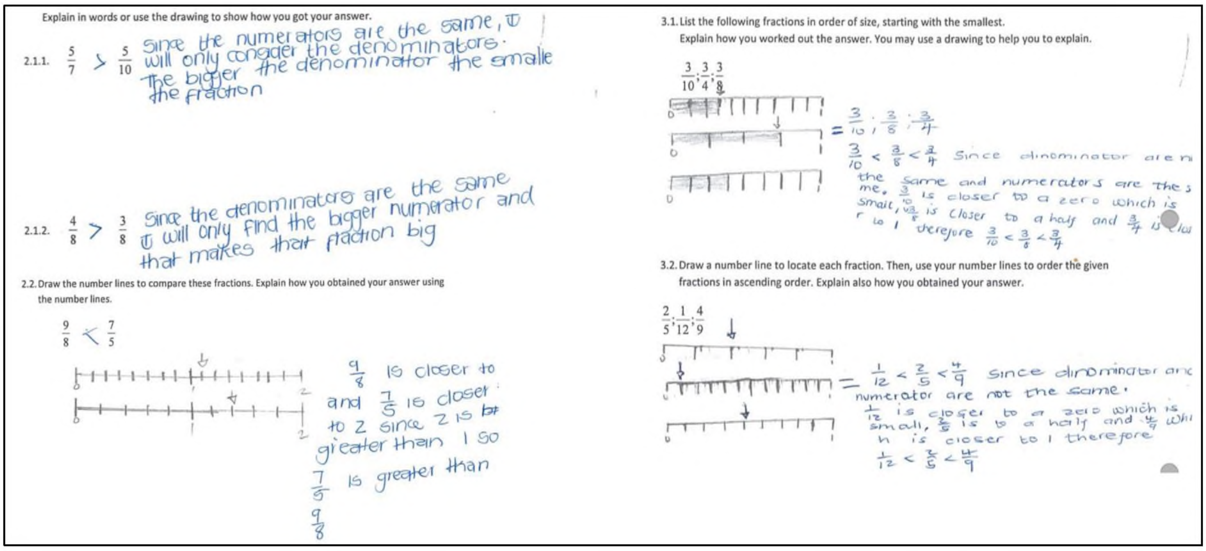


Figure 4.17: Workings of L6 and L1 (from left to right) extracted from the post-test showing the use of appropriate procedures for comparing and ordering fractions

4.4.2.1 Appropriate procedures used for comparing and ordering fractions

This subsection presents and discusses four appropriate procedures used for comparing and ordering fractions as identified from the analysis of learners' responses in the post-test, post-interviews and recall interviews. These procedures are represented in Table 4.7 by one of these keys: *EF.bars, EF.bars+Rules, E#.lines, E#.lines+Rules, E#.lines+B, Rules and Benchmarking.*

Table 4.7: The distribution of procedures used for comparing and ordering fractions in the post-test

The frequency of procedures used for comparing and ordering fractions

Procedural skill tested	Types of fractions per question in the post-test	Responses of learners											
		L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Comparing two fractions	2.1.1 Fractions with same numerator	EF.bars	E#.lines	EF.bars + Rules	Rules	Used D	Rules	Rules	Rules	EF.bars	Rules	Rules	E#.lines
	2.1.2 Fractions with same denominator	Rules	E#.lines	EF.bars + Rules	Rules	E#.lines	Rules	E#.lines + Rules	Rules	EF.bars	Rules	Rules	E#.lines
	2.2 Improper fractions with different numerators and denominators	E#.lines + B	E#.lines + B	E#.lines	Used D	Used D	E#.lines + B	E#.lines	E#.lines	E#.lines + B	E#.lines + B	E#.lines	E#.lines + B
Ordering three fractions	3.1 Fractions with same numerator	E#.lines + B	E.circles + B	EF.bars	EF.bars + Rules	Used D	Rules	E#.lines + Rules	E#.lines	E#.lines	E#.lines	E#.lines + Rules	E#.lines
	3.2 Fractions with different numerators and denominators	E#.lines + B	E#.lines + B	E#.lines + B	EF.bars	Used D	E#.lines + B	B	E#.lines	E#.lines	E#.lines	E#.lines	E#.lines

Keys	
EF.bars	The learner used *equal fraction bars* for comparing and ordering fractions appropriately
E#.lines	The learner used *equal number lines* for comparing and ordering fractions appropriately
Rules	The learner used *rules* for comparing and ordering fractions with same numerator and fractions with same denominator
EF.bars + Rules	The learner compared and ordered fractions appropriately using a combination of *equal fraction bars* and *rules* for comparing fractions with same numerator and fractions with same denominator
E#.lines + Rules	The learner compared and ordered fractions appropriately using a combination *of equal number lines* and *rules* for comparing fractions with same numerator and fractions with same denominator
B	The learner used *benchmarking* for comparing and ordering fractions appropriately
E#.lines + B	The learner compared and ordered fractions appropriately using a combination of *equal number lines* and *benchmarking*
E.circles + B	The learner compared and ordered fractions appropriately using a combination of *equal circles* and *benchmarking*
Used D	The learner compared and ordered fractions inappropriately using the sizes of denominators such that the fraction with a bigger denominator is always the bigger one

Hint: Grey shading indicates the use of inappropriate procedures for comparing and ordering fractions

4.4.2.1.1 Using equal number lines

This procedure involved representing the given fractions as the shaded length on distinct, equal number lines and comparing the fractions according to the shaded length for the respective fractions in such a way that the bigger shaded length represents the bigger fraction, while the smaller shaded length represents the smaller fraction. This procedure is represented in Table 4.7 by the key: *E#.lines*. Table 4.7 shows that this procedure was used by nine learners, namely: four learners (L2, L3, L5 and L10) from Group A and five learners (L7, L8, L9, L11 and L12) from Group B. This procedure was used to compare and order fractions with same numerator, fractions with same denominator and fractions with different numerators and denominators. For instance, in questions 2.1.1, 2.1.2, 2.2, 3.1 and 3.2 of figure 4.17, using equal number lines, learners concluded that 5/7>5/10; 4/8>3/8; 9/8<7/5; 3/10<3/8<3/4; and 1/12<2/5<4/9.

4.4.2.1.2 Using equal fraction bars

This procedure involved representing the given fractions as a shaded length on distinct, equal fraction bars and comparing fractions according to the shaded length for the respective fractions, in such a way that the bigger shaded length represents the bigger fraction, while the smaller shaded length represents the smaller fraction. This procedure is represented in Table 4.7 by the key: *EF.bars*. Table 4.7 shows that this procedure was used by four learners, namely: two learners (L3 and L4) from Group A and two learners (L1 and L9) from Group B. These learners used equal fraction bars to compare and order fractions with the same numerator, fractions with same denominator and fractions with different numerators and denominators. For instance, in questions 2.1.1, 2.1.2, 3.1 and 3.2 of Figure 4.17, these learners used this procedure to conclude that 5/7>5/10; 4/8>3/8; 3/10<3/8<3/4; and 1/12<2/5<4/9. In the post-interview, L3 used this procedure to compare and order 3/5, 4/9 and 2/7 in ascending order (see Appendix G: lines 83-84).

4.4.2.1.3 Using rules for comparing fractions with the same numerator or denominator

This procedure involved using two rules for comparing and ordering fractions with same numerator and fractions with same denominator. The first rule is: *"When fractions have the same numerator, the fraction with the smallest denominator is the biggest one, while the fraction with the biggest denominator is the smallest one"*. The second rule is that *"When fractions have the same denominator, the fraction with the biggest numerator is the biggest one, while the fraction with the smallest numerator is the smallest one"*. This procedure is represented in Table 4.7 by three keys: *Rules, EF.bars+Rules and E#.lines+Rules.* Table 4.7 shows that this procedure was used by eight learners, namely: three learners (L3, L4 and L10) from Group A and five learners (L1, L6, L7, L8 and L11) from Group B. Some of these learners (L3, L4, L7 and L11) used a combination of fraction bars (or number lines) and rules for comparing and ordering fractions as shown in Table 4.7. An example of a learner's responses using this procedure is shown in Figure 4.18.

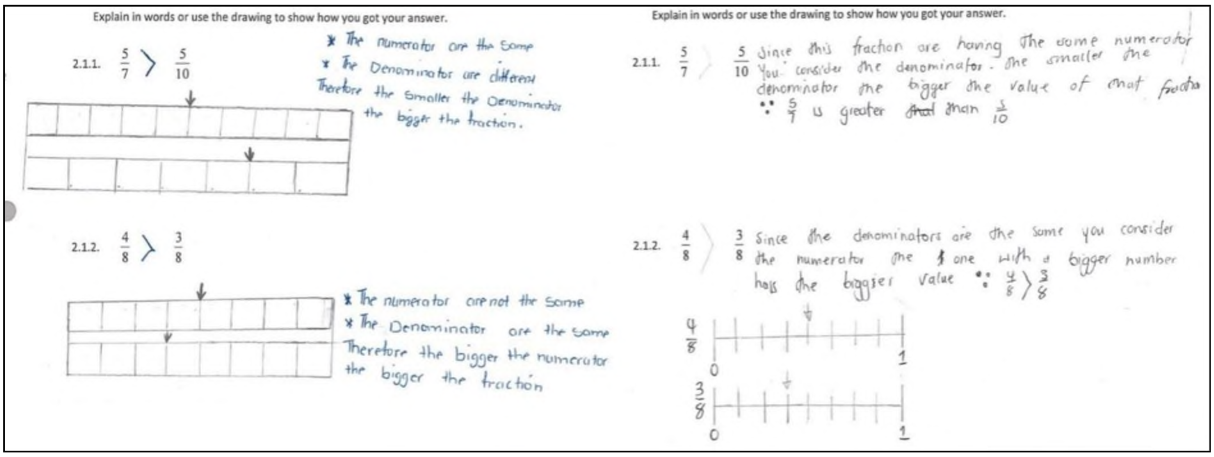
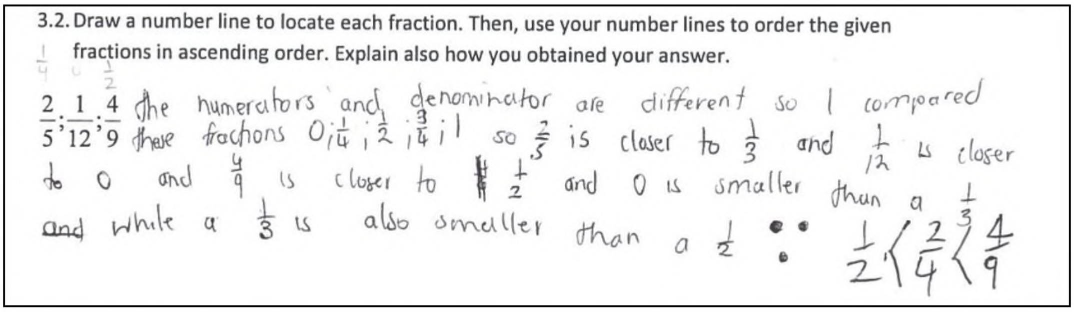


Figure 4.5: Workings of L3 and L7 (from left to right) extracted from the post-test showing the use of both the fraction bars (or number lines) and rules for comparing and ordering fractions

4.4.2.1.4 Using benchmarking

This procedure of benchmarking involves comparing and ordering the given fractions according to their closeness to the fixed reference points (0, 1/4, 1/2, 2/3, 3/4 and 1) on the number lines. This procedure is represented in Table 4.7 by two keys: *B* and *E#.lines+B*. Table 4.7 shows that this procedure was used only to compare fractions with different numerators and denominators by eight learners, namely: three learners (L2, L3 and L10) from Group A and five learners (L1, L6, L7, L9 and L12) from Group B. Examples showing the use of this procedure are shown in Figures 4.17 and 4.19. These learners used this procedure in questions 2.2 and 3.2 of Figures 4.17 and 4.19 to conclude that 9/8<7/5 and 1/12<2/5<4/9 respectively. For instance, in question 3.2, L1 explained in writing during the post-test that *"1/12<2/5<4/9 since ... 1/12 is closer to zero which is small, 2/5 is closer to a half [1/2] and 4/9 is closer to one, therefore 1/12<2/5<4/9"*. In the post-interview, L3 used benchmarking for comparing 5/8 and 3/10 by representing fractions on distinct, equal number lines. L3 explained that *"five eighths is greater than three tenths, because five eighths is closer to one, while three tenths is closer to zero"* (Appendix G: lines 117-118). In the post-interview, L6 also used equal number lines and benchmarking for comparing and ordering 3/5, 4/9 and 2/7. L6 explained that *"three fifths is closer to a whole which is one, four ninths is closer to half and two sevenths is closer to zero. ... Since zero is the smallest and then comes a half and the whole, so they will just follow each other in the sequence. The smallest will be two sevenths and then four ninths and then three fifths"* (Appendix G: lines 162-166).


3.2. Draw a number line to locate each fraction. Then, use your number lines to order the given fractions in ascending order. Explain also how you obtained your answer.

$\frac{2}{5};\frac{1}{12};\frac{4}{9}$ the numerators and denominator are different so I compared these fractions $0; \frac{1}{4}; \frac{1}{2}; \frac{3}{4}; 1$ so $\frac{2}{5}$ is closer to $\frac{1}{3}$ and $\frac{1}{12}$ is closer to 0 and $\frac{4}{9}$ is closer to $\frac{1}{2}$ and 0 is smaller than a $\frac{1}{3}$ and while a $\frac{1}{3}$ is also smaller than a $\frac{1}{2}$ $\therefore$ $\frac{1}{2}<\frac{2}{4}<\frac{4}{9}$

Figure 4.19: Workings of L7 extracted from the post-test showing the use of benchmarking for comparing and ordering fractions with different numerators and denominators

4.4.2.2 Inappropriate procedure used for comparing and ordering fractions

This subsection presents and discusses one inappropriate procedure used for comparing and ordering fractions as identified from the analysis of learners' responses in the post-test and post-interviews. This procedure involved using the sizes of only the denominators to compare and order fractions. This procedure is represented in Table 4.7 by the key: *Used D*. Table 4.7 shows that this procedure was used by two learners (L4 and L5) from Group A. They compared and ordered fractions in such a way that the fraction with the bigger denominator is the bigger fraction, while the fraction with the smaller denominator is the smaller one. For instance, in questions 2.1.1, 2.2, 3.1 and 3.2 of Figure 4.17 (and see Table 4.7), L4 concluded that 5/7<5/10; 9/8>7/5; 3/4<3/8<3/10; and 2/5<4/9<1/12. In the post-interview, L5 used this procedure to conclude that 5/8<3/10 and 3/5<2/7<4/9 (see Appendix G: lines 92-93).

4.4.3 Adding two fractions

This subsection presents and discusses changes in learners' conceptual understanding of and procedural fluency with the procedures they used to add two fractions that were evident after the intervention. These changes for adding fractions were assessed in section 4 of the post-test and post-interviews. This section consisted of three questions and an example showing a learner's responses to the questions is shown in Figure 4.20.

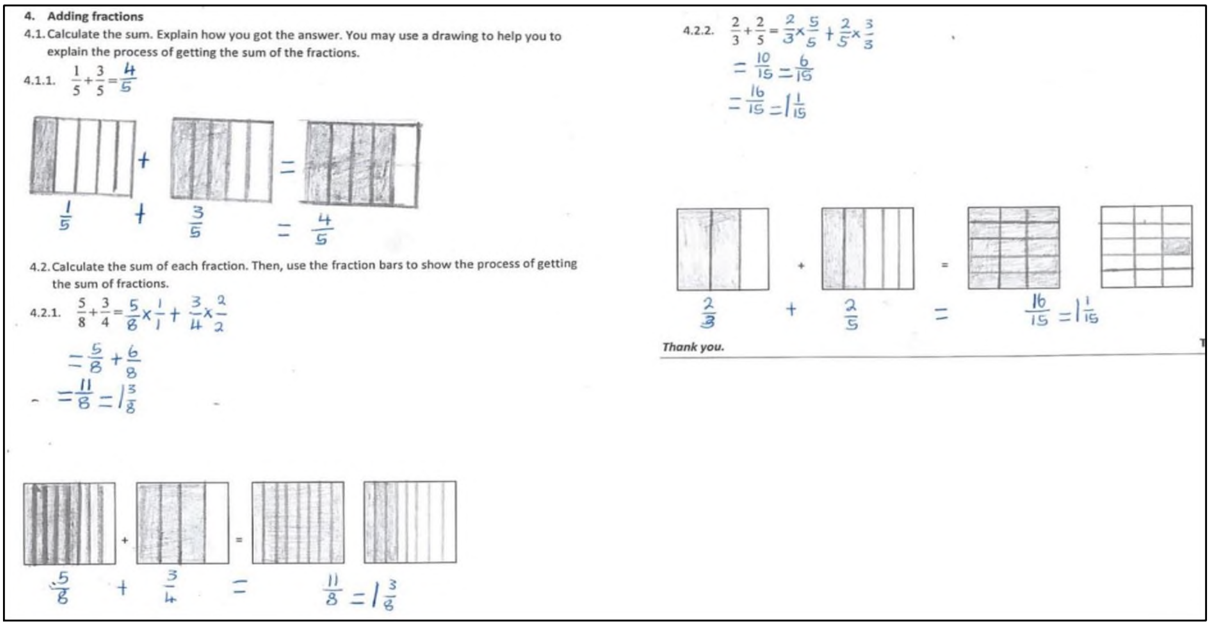


Figure 4.20: Workings of L9 extracted from the post-test showing the use of fraction bars to represent visually the lowest common denominator method for adding fractions

The next two subsections 4.4.3.1 and 4.4.3.2 present and discuss the appropriate and inappropriate procedures used for adding two fractions after the intervention.

4.4.3.1 Appropriate procedures used for adding two fractions

This subsection presents and discusses two appropriate procedures used for adding fractions as identified from the analysis of learners' responses in the post-test and post-interviews. These procedures are represented in Table 4.8 by two keys: *PF* and *CU*.

Table 4.8: The distribution of procedures used for adding two fractions in the post-test

The frequency of procedures used for adding two fractions													
Procedural skills tested	**Types of fractions per question in the post-test**	**Responses of learners**											
		L1	**L2**	**L3**	**L4**	**L5**	**L6**	**L7**	**L8**	**L9**	**L10**	**L11**	**L12**
Adding two fractions	4.1.1 Fractions with same denominator	CU	IARM	PF	PF	ANDT	CU	CU	CU	CU	CU	CU	CU
	4.2.1 Fractions with different numerators and denominators	CU	PF	ANDT	No workings	ANDT	CU	CU	CU	CU	ANDT	CU	CU
	4.2.2 Fractions with different numerators and denominators	CU	PF	ANDT	No workings	ANDT	CU	CU	CU	CU	ANDT	CU	CU
Keys													
PF	The learner added fractions appropriately using the lowest common denominator method only												
CU	The learner displayed appropriate conceptual understanding of and procedural fluency of adding fractions by visually representing the lowest common method for adding fractions using equal fraction bars												
ANDT	The learner added fractions inappropriately by adding numerators together and adding denominators together												
IARM	The learner added fractions using inappropriate arithmetic manipulation (*IARM*)												
No workings	The learner showed no workings for adding fractions												
Hint: Grey shading indicates the use of inappropriate procedures for adding fractions													

4.4.3.1.1 Using the lowest common denominator method for adding fractions

This procedure involved using the lowest common denominator method for adding two fractions with the same denominator and fractions with different denominators. This procedure is represented in Table 4.8 by the key: *PF*. Table 4.8 shows that this procedure was used by three learners from Group A, namely: L2, L3 and L4. These learners did not use fraction bars to visually represent the lowest common denominator method for adding fractions. An example showing workings of a learner's responses using this procedure is shown in Figure 4.21.

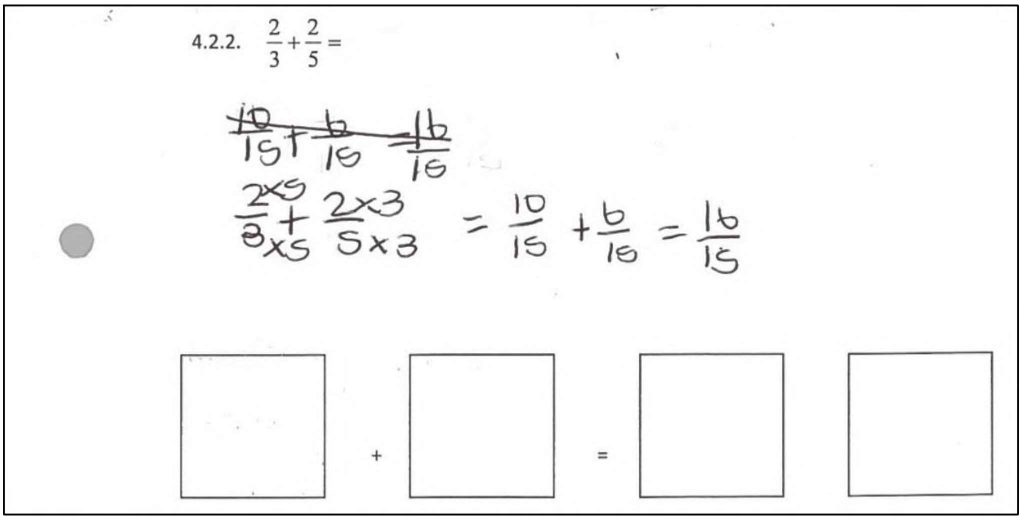


Figure 4.21: Workings of L2 extracted from the post-test showing the use of the lowest common denominator method for adding two fractions

4.4.3.1.2 Using the fraction bars to visually represent the lowest common denominator method for adding fractions

This procedure involved using equal fraction bars to visually represent the lowest common denominator method for adding fractions with the same denominator and fractions with different denominators. This procedure is represented in Table 4.8 by the key: *CU.* Table 4.8 shows that this procedure was used by all seven learners from Group B, namely: L1, L6, L7, L8, L9, L11 and L12. An example showing a learner's responses using this procedure is shown in Figure 4.20. Figure 4.20 shows that L9 used fraction bars to show that 1/5+3/5=4/5; 5/8+3/4=5/8+6/8=11/8=1 3/8; and 2/3+2/5=10/15+6/15=16/15=1 1/15. These learners used a combination of equal fraction bars and the lowest common denominator method to add fractions and this suggests that these learners displayed both conceptual and procedural engagement with fraction models and fraction symbols. In the post-interview, this dual engagement was displayed by L6 to show that 2/5+2/3=6/15+10/15=16/15=1 1/15 (see Figure 4.20 and Appendix G: lines 138-154).

4.4.3.2 Inappropriate procedures used for adding fractions

This subsection presents two inappropriate procedures used for adding fractions with same denominator and fractions with different denominators, as identified from the analysis of learners' responses in the post-test and post-interviews. These procedures are represented in Table 4.8 by two keys, namely: *ANDT* and *IARM.* The analysis of learners' responses in the post-test and post-interviews showed that every learner who used these inappropriate procedures to add two fractions did not visually represent the lowest common denominators using fraction bars.

4.4.3.2.1 Adding numerators together and denominators together

This inappropriate procedure involved finding the sum of two fractions by adding numerators together and adding denominators together. This procedure is represented in Table 4.8 by the key: *ANDT*. Table 4.8 shows that this procedure was used by three learners from Group A, namely: L3, L5 and L10. These learners used this procedure to add fractions with same denominator and fractions with different denominators. An example showing a learner's responses using this procedure is shown in Figure 4.10. For instance, in questions 4.1.1, 4.2.1 and 4.2.2 of Figure 4.20, these learners used this procedure to conclude that 1/5+3/5=4/10; 5/8+3/4=8/12; and 2/3+2/5=4/8. The use of this inappropriate procedure for adding fractions suggests that these learners showed lack of conceptual understanding of and procedural fluency with the appropriate procedures used for adding fractions. In the post-interviews, L3 and L5 used this procedure to show that 2/5+2/3=4/8 (see Appendix G: lines 93-94 and 120-121).

4.4.3.2.2 Using inappropriate arithmetic manipulation

This procedure involved using inappropriate arithmetic manipulation to add fractions. This procedure is represented in Table 4.8 by the key: *IARM*. Table 4.8 shows that this procedure was used only by one learner from Group A, namely: L2. An example showing the workings of L2 using this procedure is shown in Figure 4.22. The use of this inappropriate procedure showed little developing conceptual understanding or procedural fluency for adding fractions over the intervention period.

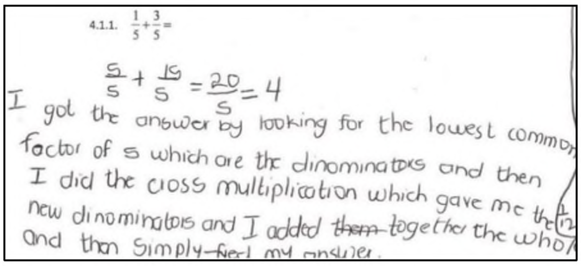


Figure 4.22: Workings of L4 extracted from the post-test showing the use of inappropriate arithmetic manipulation of the procedures for adding fractions

4.5 CONCLUSION

This chapter presented and discussed in section 4.2, the nature of learners' conceptual understanding of and procedural fluency of fractions before the intervention. Section 4.3 presented and discussed the changes in learners' conceptual understanding and procedural fluency of fractions that were evident after the intervention. The analysis of both sections were presented in three subsections, namely: relating unit subdivisions to the denominator of fractions; procedures used for comparing and ordering fractions; and procedures used for adding two fractions. For each of the subsections, examples of inappropriate and appropriate responses were identified and related the description of learners' conceptual understanding of and procedural fluency with fractions before and after the intervention. The analysis of learners' responses in the post-test and post-interviews showed that learners from Group B changed much more during the period of the intervention, than learners from Group A. They showed developing conceptual understanding of and procedural fluency with fractions less than one whole unit; fractions greater than one whole unit; comparing, ordering and adding fractions as shown in Table 4.1.

CHAPTER 5

DATA PRESENTATION PART 2

5.1 INTRODUCTION

This chapter presents the teaching process of the intervention in this research project. The data presented in this chapter are drawn from two research instruments: transcripts of lesson videos and learners' completed worksheets.

5.2 THE TEACHING PROCESS OF THE INTERVENTION

The intervention consisted of nine lessons over the period of nine days (see table 3.5) and thirteen worksheets were used (see tables 3.1-3.4). The teaching in the intervention used number lines and area models to develop conceptual understanding of fractions. Learners were encouraged to read fractions using appropriate names, and to recognise and use fraction notation. The area models and number lines were used for comparing and ordering fractions, to encourage the use of benchmarking and appropriate rules for comparing fractions. In addition, the area models were used to visually represent the lowest common denominator method for adding two fractions. The analysis of the teaching intervention is presented in five subsections, namely: subsections 5.2.1-5.2.5. Each subsection presents and discusses one of five themes, namely:

- Reading fractions using appropriate names;
- Developing conceptual understanding of fraction denominators by writing whole numbers as fractions with unit subdivisions;
- Differentiating fractions less than one unit from fractions greater than one unit;
- Using benchmarking and rules for comparing and ordering fractions;
- Visually representing the lowest common denominator method for adding two fractions.

The first three themes listed above, are related to the process for developing conceptual understanding of fractions as numbers, while the last two themes are related to the process for developing conceptual understanding and procedural fluency when comparing, ordering and adding fractions. The process for developing conceptual understanding of fractions as numbers was a cumulative, linear process with five key activities, which are presented in Figure 5.1.

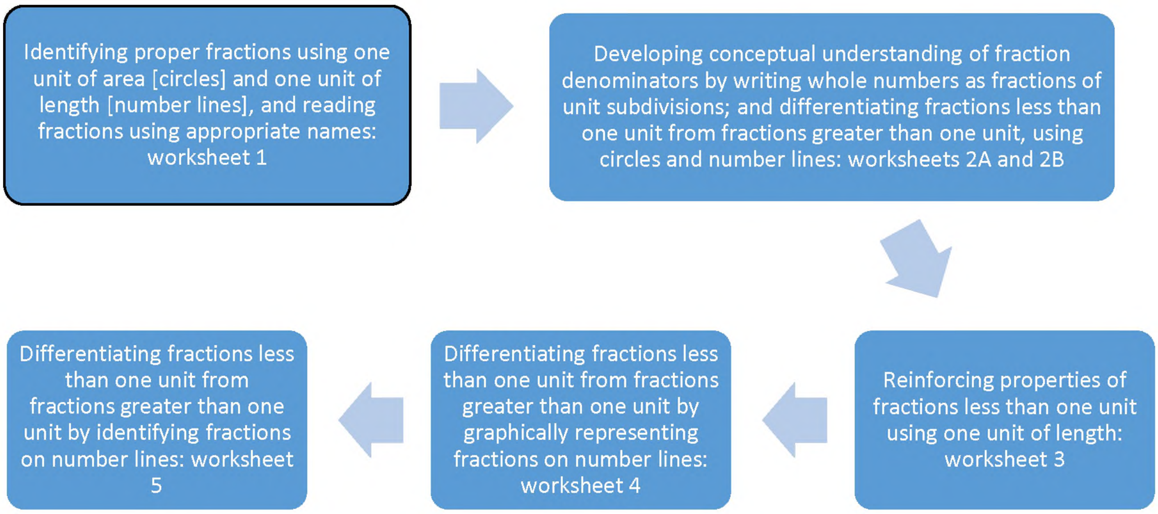


Figure 5.1: The cumulative, linear process for developing learners' conceptual understanding of fractions during the intervention

5.2.1 Reading fractions using appropriate names

This subsection presents the learning process involving activities that enabled learners to read fractions using appropriate names that related the unit subdivisions to the denominator of the fractions. Reading fractions using appropriate names was first introduced to the learners during the first lesson, and was reinforced continuously from the second until the last lesson of the intervention. This discussion will concentrate on events that happened from the first until the fourth lessons of the intervention, where most of the learners' tasks focused on developing a conceptual understanding of fractions.

The first lesson made use of worksheet 1 (see appendix A), which was composed of single pre-partitioned models, namely: a number line one unit long and a single circle that was partly shaded. Therefore, all names that were learned during first lesson were for fractions less than one unit. The learning objectives for the first lesson were threefold. Firstly, it was expected that learners use appropriate fraction notation to identify the size of the area shaded in the given model. Secondly, learners were expected to read fractions using appropriate names that related the unit subdivisions to the fraction denominators. Thirdly, learners were expected to recognise that subdivisions in a whole unit should be equal, and that counting them allowed the fraction to be described as a single number. The next paragraphs present four extracts (5.1-5.4) of interactions during the first four lessons of the intervention.

During the first lesson, the teacher used the first item of worksheet 1 to demonstrate to the class how to read fractions using appropriate names. He started by asking the class to identify the fraction shown in the first item of worksheet 1. Extract 5.1 shows a snapshot of their interaction:

Extract 5. 1: Extract from the transcript of the introduction of worksheet 1

T: Now in this activity [worksheet 1], I would like you to write fractions in words. For example, 3/4 as three quarters [he wrote both 3/4 and 'three quarters' on the chalkboard].
T: What is the number of shaded parts here [item one shows a fraction 2/3]?
Ls: Two.
T: And what is the number of total parts in the whole unit?
Ls: Three.
T: Ok. [2/3 was recorded on the chalkboard].
T: Very good, and to write this fraction in words?
L1: Two over three.
T: Two over three or...?
L12: Two thirds
T: Good. Avoid the use of the word 'over' when describing fractions, because a fraction is one number and not two different numbers (Appendix I: lines 60-68, 71-75).

Thereafter, learners were instructed to complete the rest of the items on worksheet 1 using the first worked example. Learners worked collaboratively in groups of three to complete the worksheet. Firstly, they were expected to identify the appropriate fraction notation for the area shaded in a given model, and then write down appropriate fraction names for the fraction notation they had produced. The teacher observed that all the groups related the number of unit subdivisions shaded to the fraction numerators and the number of subdivisions in a unit to the fraction denominators appropriately, based on their written work.

The extracts of learners' written responses from worksheet 1 showed that these learners used appropriate names for all ten items. These included: two thirds (2/3); four sixths (4/6); three fifths (3/5); five eighths (5/8); three sixths (3/6); six ninths (6/9); one fifteenth (1/15); six tenths (6/10); four sevenths (4/7); and six elevenths (6/11). While most groups gave appropriate names, three groups out of four groups used inappropriate names to describe 1/15 as *"one over fifteen"* or *"fifteenths"*. After the first group failed to read the name correctly, all groups were asked to read what they have written down for 1/15. Surprisingly, it was only the fourth group which had the correct name of *"one fifteenth"*. This was accepted by the class after the teacher explained that the fraction name should be read as one fifteenth since only one part was shaded and the –s should only be added to show that more than one part is shaded.

Mastering the reading of fractions using appropriate names did not appear to have been attained in the first lesson. During the second lesson, learners were introduced to proper fractions and improper fractions. Here, some learners were still using inappropriate names while some of the learners started using appropriate names. Extract 5.2 shows an interesting interaction from the second lesson.

Extract 5. 2: Extract from the transcript of the second lesson

L9: Sir?
T: Yes.
L9: How did you get eleven over four [1 1/4]?
T: Say that again? You meant how did I get one and a quarter?
L9: Yes, sir! …
T: What did you write here?
L7: Eleven over twelve.
T: Ooh, are you sure?
T: What fraction is that?
L6: Three over four.
T: Three over four?
L6: Three quarters.
T: *He*, it is like your fraction represents two separate things. Okay, three quarters, *ne* [right]?
L6: Yes, sir (Appendix I: lines 104-108, 128-130 and 177-182).

The learners that read fractions using appropriate fraction names during the second and third lessons includes L1, L5, L7, L10, and L12. Extract 5.3 shows one of the incidents.

Extract 5.3: Extract from the transcript of the feedback session of second and third lessons

T: What is the answer…?
L1: Six tenths.
T: Thank you. …
T: What is the answer in b?
L5: One fifth.
T: Do you agree?
Ls: Yes.
T: Number three?
L7: Six eighths.

T: Do you agree?
Ls: Yes!
T: Number four?
L10: Two sixths.
T: Do you agree?
Ls: Yes.
T: Number five?
L12: Six ninths. (Appendix I: lines 515-516 and 524-537)

During the fourth lesson, some learners were still reading fractions inappropriately. Extract 5.4 shows one of the incidents.

Extract 5.4: Extract from the transcript of the fourth lesson

T: What is that mixed number?
L8: Two whole and five out of ten.
T: Yes, two and five tenths, *ne*?
Ls: Yes. (Appendix I: lines 597-600)

These snapshots showed that some learners were starting to master reading fractions using appropriate names while others were still learning this practice. The next subsection presents how the intervention enabled learners to develop conceptual understanding of the fraction denominator.

5.2.2 Developing conceptual understanding of fraction denominators by writing whole numbers as fractions with unit subdivisions

One of the difficulties learners showed before the intervention was to recognise whole numbers on the number lines as fractions with unit subdivisions. This subsection presents the process of interaction that enabled learners to develop a conceptual understanding of the denominator of fractions and to recognise whole numbers as fractions with unit subdivisions, using circles and number lines. It was expected that learners would recognise a fraction denominator as a symbolic representation of the number of unit subdivisions that make up one whole unit. Based on this expectation, it was also expected of learners to recognise and express whole numbers (on number lines) as fractions with unit subdivisions – fractions whose numerators are multiples of their denominators.

The relation between whole numbers and unit subdivisions appeared to be one of the fundamental keys that contributed to learners' changes presented in Chapter Four. Much of the teaching on writing whole numbers as fractions in the first four lessons, focused on helping learners to understand the first unit as a fraction with unit subdivisions. This is shown by the number of extracts presented in this subsection.

After helping learners to write the first unit as a fraction with unit subdivisions, learners were helped to identify other whole numbers as fractions with unit subdivisions. This subsection presents interactions from the first four lessons of the intervention, which made use of worksheets 1, 2A, 2B, 3, 4 and 5 (see Table 3.5). These worksheets were designed to develop learners' experience in identifying fractions and representing fractions using number lines and circles.

The following extracts 5.5-5.7 show interactions during the first lesson that indicate how two fractions less than one (3/4 and 7/10) were used to demonstrate to the class that a fraction denominator is a symbolic representation of subdivisions in the first unit. The teacher drew two fraction bars on the chalkboard. The first fraction bar was fully shaded and he explained to the class that it represented one unit like a loaf of bread. The second fraction bar had the same width as the first fraction bar but with a shorter length. The teacher asked the class to estimate the size of the second fraction bar in comparison to the first fraction bar. Extracts 5.5-5.7 show the details.

Extract 5.5: Interaction on reading fraction bars to develop conceptual understanding of fraction denominators during the first lesson

L7: Seven over ten.
T: Or?
L2: Three over four.
T: These numbers are almost giving the same quantity. Let's look at these two fractions 'seven over ten' and 'three over four'. What does ten or four represent?
L8: Four represent a quarter. (Appendix I: lines 7-12)

The teacher wrote one over four on the chalkboard and asked the class to confirm whether one over four was a quarter. Extract 5.6 shows the details.

Extract 5.6: Interaction on reading fraction bars to develop conceptual understanding of fraction denominators during the first lesson

Ls: Yes.
T: What does four tell you?
L5: It is the number of pieces.
T: The number of pieces where?
L6: The number of pieces in which a whole is divided.

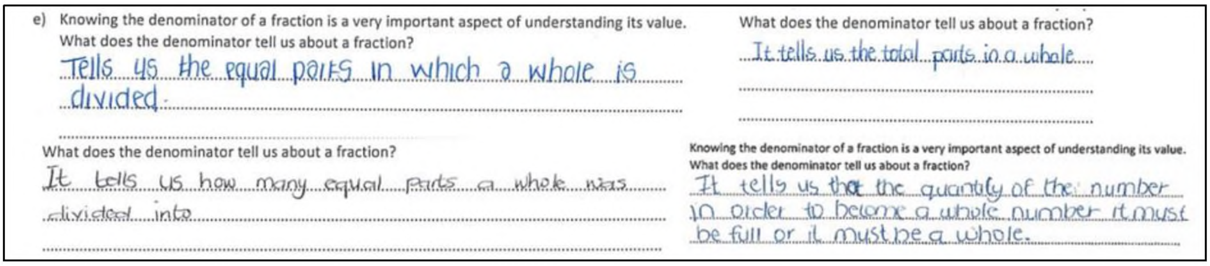
T: Okay... But what does four tell us? It tell us the number of pieces that make up one whole unit. And how many pieces does this whole unit have?
Ls: Four out of four.
T: How much?
Ls: Four over four. (Appendix I: lines 15-21)

The teacher was satisfied that learners were beginning to think of one unit [a fraction bar] as a fraction with unit subdivisions (i.e. 1=4/4).He drew a number line on the chalkboard and marked the position of zero, one and two. He asked the class to locate 7/10 on the number line by estimating. L3 went to the chalkboard and wrote seven on the number line between zero and one but closer to one. Extract 5.7 shows the details.

Extract 5.7: Interaction on reading of number lines to develop conceptual understanding of fraction denominators during the first lesson

T: What is that you wrote?
L3: Seven.
T: Look at the given fraction, is that seven? [L3 erased 7 and wrote 7/10 correctly]
T: As you can see, the denominator is bigger than the numerator, what does that mean? If the fraction's denominator is bigger than the numerator then the fraction is less than?
Ls: Ten.
T: Less than?
Ls: Ten.
T: How can you write one as a fraction with a denominator of ten?
L12: You divide ten by ten.
T: Very good. (Appendix I: lines 28-38)

Another interesting moment occurred during the first lesson, when learners were asked to share with the class their groups' answers to the following question in worksheet 1: "What does the denominator tell us about the fraction"? At first, these learners did not make sense of the question. The teacher repeatedly explained the question to the class until he decided to give the answer to the class. The class was then asked to write down the answer to the question that they had heard from the teacher. Figure 5.2 shows the groups' record of what they had heard from the teacher. Their responses in Figure 5.2 show that learners understood the conceptual meaning of fraction denominators as unit subdivisions that make up one whole unit.


e) Knowing the denominator of a fraction is a very important aspect of understanding its value. What does the denominator tell us about a fraction?
Tells us the equal parts in which a whole is divided.

What does the denominator tell us about a fraction?
It tells us the total parts in a whole.

What does the denominator tell us about a fraction?
It tells us how many equal parts a whole was divided into

Knowing the denominator of a fraction is a very important aspect of understanding its value. What does the denominator tell us about a fraction?
It tells us that the quantity of the number in order to become a whole number it must be full or it must be a whole.

Figure 5.2: Workings of the four groups of learners showing learners' responses to the question for the conceptual meaning of the fraction denominator

The next three extracts 5.8-5.10 show interactions during the third and fourth lessons respectively, on how conceptual understanding of the fraction denominator was reinforced. During the third lesson, the teacher probed L1 to explain the meaning of a quarter [for 3/4]. Extract 5.8 shows the details.

Extract 5.8: Interaction on developing conceptual understanding of a quarter during the third lesson

T: And the quarter tells us how many times a circle should be divided into. You divide a circle into how many parts?
L1: Four parts.
T: Why four parts?
L1: Because the denominator is four.
T: And you got the denominator from where?
L1: From the shaded parts.
T: The shaded parts of what? Of a circle or of a number line?
L1: From the equal parts of the number line.
T: Okay. How many parts is one divided into, on a number line?
L1: Four.
T: Four, *ne*?
L1: Yes!
T: Okay, thank you very much (Appendix I: lines 184-197).

The next two extracts 5.9 and 5.10 show how L5 and L2 grappled to identify fraction denominators by counting unit subdivisions on the number lines and circles. During the fourth lesson the teacher demonstrated to L5 how to use denominator of 5/8 to partition the unit length on the number line. Extract 5.9 shows the details.

Extract 5.9: Interaction from the transcript of the fourth lesson on L5 using the denominator to partition the number line

T: Okay, the denominator tell you what about the division of the number line? You need to count the divisions from zero like one two three four five six seven eight and where there is eight, you put one.
L5: Okay sir.
T: And eight is the denominator.
L5: Okay.
T: Cool. (Appendix I: lines 561-567)

During the third lesson, the teacher asked L2 to explain her answers shown in Figure 5.3. Extract 5.10 shows the details of how the teacher assisted L2 to write whole numbers as fractions with unit subdivisions.

Extract 5.10: Interaction from transcript of the third lesson on L2 writing whole numbers as fractions with unit subdivisions

T: Three over three is what? Three into three, goes how many times?
L2: Six.
T: Three into three?
L2: One.
T: Now, what fraction is here [at 2]? Write two as a fraction using these divisions.
L2: Two over two.
T: Two over two is one.
L2: Yes!
T: No! If you said two over two, then it will give you one [2/2=1]. So what fraction is here?
L2: One.
T: No, I want you to use these divisions [on a number line]. It is not one. One is a whole number here. What fraction is here? You start … one third two thirds three thirds four thirds five thirds and here [at two]?
L2: A quarter maybe.
T: No! A quarter *shike* [how]?
L2: Ooh!
T: Just continue counting.
L2: Five thirds, six thirds.
T: Write down for me six thirds here [*L2 wrote 6/3 at 2]*.
T: Now simplify. The answer is going to be what? Three into six goes how many times?
L2: Two times.
T: Yes. That's what I wanted you to see. That's why six thirds is two.
T: Now, I want you to write three, using the same divisions.
L2: You meant to continue counting up to three?
T: Yes!
L2: Nine.
T: Yes. That is if you have nine parts divided into three, you will get?

L2: Three.
T: Yes. And if you have 12/3, you will get?
L2: Four.
T: Yes. That's where the thing is coming from.
L2: Yes, sir (see Appendix I: lines 301-329).

Figure 5.3 shows that L2 identified the fractions of area shaded for circles as 5/3 = 1 2/3 and 12/5 = 2 2/5. This change can be attributed to the ability of the learner to think of whole numbers as fractions with unit subdivisions.

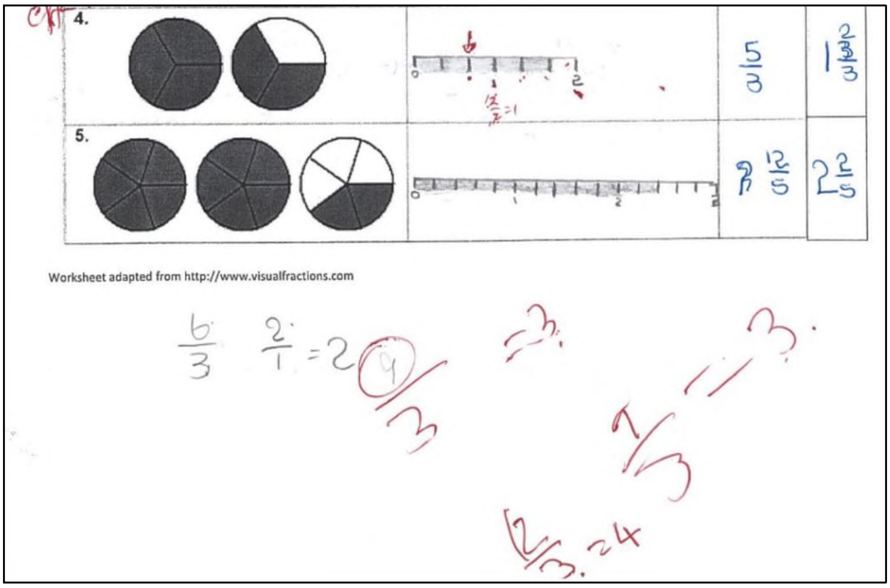


Figure 5.3: Workings of L2 extracted from worksheet 2B on expressing whole units as fractions of unit subdivisions

By end of the fourth lesson, learners were able to relate unit subdivisions to fraction denominators as expected of them. Extract 5.11 shows the details.

Extract 5.11: Interaction during the fourth lesson on learners relating unit subdivisions to fraction denominators

T: By looking at this fraction (1 1/3), you should be able to tell how the space between two whole numbers is divided into. The space between the whole numbers is divided into?
Ls: Three parts.
T: Three parts?
Ls: Yes.
T: Okay, and for 1 ¼, the space between whole numbers should be divided into?
Ls: Four parts (Appendix I: lines 630-636).

5.2.3 Differentiating fractions less than one unit from fractions greater than one unit

This subsection presents the process that enabled learners to differentiate fractions less than one whole unit from fractions greater than one whole unit. This distinction was based on the following two definitions. Firstly, learners were expected to recognise proper fractions as the symbolic representation (measurement) of fractional quantities less than one whole unit of area or one unit of length. Using this definition, learners were expected to think of proper fractions as points less than one on the number line and to locate these fractions between zero and one. By using the area models (circles), learners were expected to relate proper fractions to the use of a single area model. Secondly, learners were expected to recognise improper fractions (mixed fractions) as the symbolic representation of fractional quantities greater than one whole unit of area and one unit of length. By applying this definition, they were expected to think of improper fractions as points greater than one and locate these fractions after one on number lines. By using area models, learners were expected to relate improper fractions to area models that showed more than one whole. This distinction was initially made during the first lesson and the discussion in this subsection is limited to the first four lessons of the intervention, which made use of worksheets 1, 2A, 2B, 3, 4 and 5.

This subsection presents relating of fractions to one whole unit in four parts. Firstly, the subsections present three extracts 5.12-5.14 on how learners learned to differentiate fractions in relation to one whole unit. Secondly, the subsection presents two extracts 5.15 and 5.16 on how learners applied conceptual understanding of proper fractions and improper fractions in relation to one to obtain appropriate responses. Thirdly, the subsection presents extract 5.17 showing how learning of differentiating fractions in relation to one whole unit was not simple, and fourthly, the subsection presents extracts of how properties of proper fractions and improper fractions in relation to one whole unit was reinforced.

The next two extracts 5.12 and 5.13 present how learners grappled to make a distinction between fractions less than one whole unit and fractions greater than one unit. In most cases, learners were asked to justify their answers by applying the appropriate properties of the fractions. Extract 5.12 presents the interaction from the first lesson showing how learners and the teacher differentiated fractions less than one unit from fractions greater than one unit by relating the fraction notation to one on the number line.

During the first lesson, the teacher first partitioned the distance between zero and one into ten equal parts and counted the parts together with learners and labelled each subdivision after zero starting with 1/10 up to 10/10 which is 1. The teacher also divided the space between one and two into tenths.

Extract 5. 12 Interaction of the first lesson showing how learners related fractions greater than one whole to one

T: What is the next fraction after one?
L6: It is 1 1/10.
T: 1 1/10 is bigger than what?
Ls: It is bigger than one whole.
T: Very good. Whenever the denominator and numerator are equal, then it is equal to one whole. It does not matter how many parts one whole unit is divided into, *ne*?
Ls: Yes, sir. (Appendix I: lines 42-48)

Thereafter, the teacher wanted to reinforce the relation between one, the fraction numerator and fraction denominator, so he introduced a different fraction 7/8 to test whether learners understood this relationship. Extract 5.13 shows the details.

Extract 5. 13: Interaction of the first lesson showing how learners related fractions less than one whole unit to one

T: If you have 7/8, is it less than one whole or more than one whole?
Ls: Less than one whole.
T: Okay. Since 7/8 is not a full whole, is this fraction bigger than one?
Ls: No.
T: Is it bigger than one?
Ls: No, sir.
T: For you to have one full whole, how many pieces do you need?
Ls: Eight.
T: Out of how many pieces?
Ls: Eight (Appendix I: lines 51-59).

The following extract 5.14 from the second lesson presents how the teacher and learners differentiated fractions less than one unit from fractions greater than one unit by relating two different fractions to one on the number line.

Extract 5. 14: Interaction from second lesson showing how learners differentiated fractions less than one whole unit from fractions greater than one whole unit

T: If I give you a fraction like 3/5, what type of fraction is 3/5?
Ls: Proper fraction.

T: Can you write a proper fraction as a mixed number?
Ls: No, sir!
T: Why? Why is it true?
L4: Because it is already in simplest form.
T: Yes, in other words?
L7: It does not have a whole number. You can only change it to an improper fraction if it has a whole number like 1 1/4.
T: Okay, what is the difference between 3/5 and 5/4? How will you tell which one is a proper fraction and which one is an improper fraction?
L3: For 5/4, the numerator is bigger than the denominator while for the proper fraction 3/5, the numerator is smaller than the denominator.
T: Okay, when you look on the number line, on which side of one will you find the improper fraction? On the left or on the right?
Ls: On the right.
T: Okay, what does this mean? It means, if the number is an improper fraction, it should be shown on the right hand side of?
Ls: One.
T: Of one, *ne*?
Ls: Yes.
T: Only numbers that are greater than one can be written as improper fractions, and if the fraction is less than one, you cannot express it as an improper fraction, *ne*?
Ls: Yes, sir!
T: Because, there is no a whole yet. But for an improper fraction which is greater than one, it can be written as an improper fraction. (Appendix I: lines 40-57)

The extract 5.14 presented the conceptual understanding for proper fractions and improper fractions in relation to one whole unit. The next two extracts 5.15 and 5.16 present how three learners applied conceptual understanding of proper fractions and improper fractions in relation to one to obtain appropriate responses.

Extract 5. 15: Interaction showing how L7 applied the relation of fractions to one to obtain the appropriate fraction

T: What did you write here?
L7: Eleven over twelve [11/12].
T: Ooh, are you sure?
L7: Mhh, yes!
T: What type of fraction is this one?
L7: It is a proper fraction.
T: Is this a proper fraction?
L7: Ooo!

T: Mhh! [L7 erased 11/12 and wrote the appropriate fraction 2 3/4 = 11/4]. (Appendix I: lines 70-72)

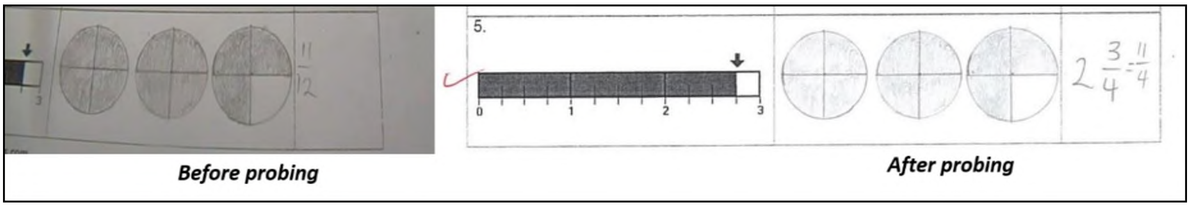


Figure 5.4: Workings of L7 extracted from worksheet 2A showing how he used the position of one on the number line to identify improper fractions appropriately during the second lesson

The process of differentiating fractions less than one unit from fractions greater than one unit was not simple. Some of the challenges involved using appropriate unit subdivisions to partition the circles. Other challenges were technical such as using the ruler, compass and protractor to partition the number lines and circles. Figure 5.5 shows a picture of two learners using a protractor and finding it difficult to partition the circles appropriately as well as the teacher checking learners' work.

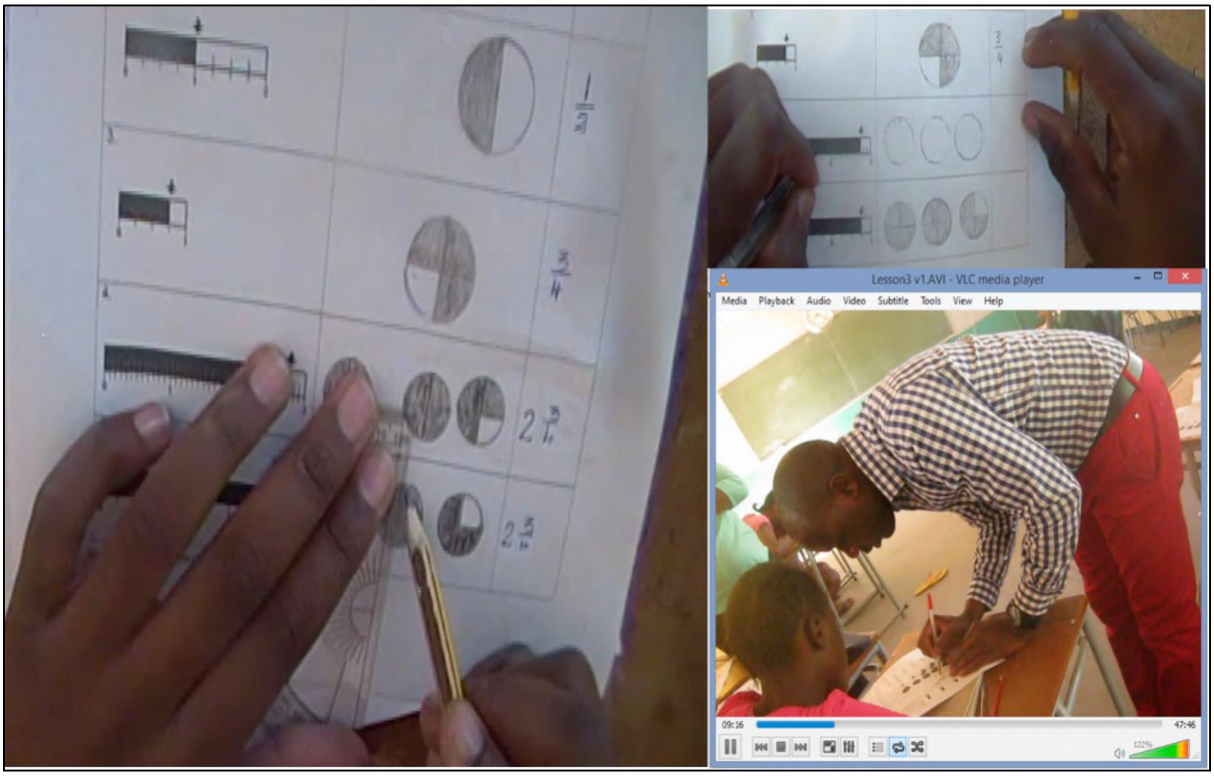


Figure 5.5: The peak of classroom interaction as learners tried to make sense of fraction models to differentiate fractions less than one unit from fractions greater than one unit during the third lesson

The first picture above, shows an extract of worksheet 2A of a learner that identified incorrect fractions ½ and 2 ¾ instead 1 and 2 9/12 respectively. The second picture showing a child's workings, the child appeared not knowing what to do to partition the circles to represent the fraction 2 9/12. Extract 5.16 presents how L4 did not differentiate fractions less than one from fractions greater than one while he was completing worksheet 2A (see Figure 5.6 for details).

Extract 5.16: Interaction showing how L4 did not differentiate fractions in relation to one whole unit

T: What is the next fraction?
L4: Eleven …
T: Eleven over?
L4: Eleven eighths.
T: … Is it eleven eighths?
L4: No, sir!
T: Okay, what is the fraction?
L4: Two whole and three become eleven.
T: Are you reading as a mixed number? Just give us one.
L4: Proper fraction.
T: Improper fraction?
L4: Yes.
T: Okay, what is your improper fraction?
L4: Three out of four. …
T: Three out of four is three quarters.
Ls: It is not three quarters, sir! …
L8: It is a mixed number sir. ….
T: A mixed number can be improper?
Ls: Yes.
T: So, it is not a proper fraction, *ne*?
Ls: Yes.
T: Why is it not a proper fraction?
L6: Because the numerator is bigger than the denominator.
T: Mhh! The number of parts shaded exceeds one as a whole unit. So, what is the answer?
L4: Three over four.
T: Who can help us to read the fraction correctly?
L2: Two and three quarters.
T: Yes. First you read the whole number and to connect them, use the word 'and'. (Appendix I: lines 222-259)

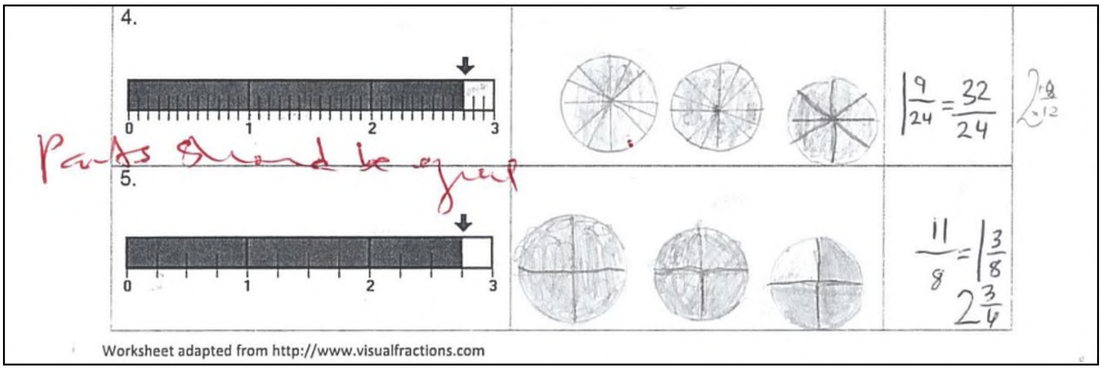


Figure 5.6: Workings of L4 extracted from worksheet 2A showing the use of inappropriate understanding to identify fractions during the second lesson

Extract 5.17 presents a combination of interaction of learners using conceptual understanding of fractions in relation to one whole unit of circles to justify their responses towards the end of the second lesson.

Extract 5. 17: Interaction of the second lesson showing how learners related the identified fraction to one whole unit of circles

T: What is the fraction for number four?
L10: It is thirty three twelfths.
T: Thank you very much. And how many circles should you have?
Ls: Three.
T: And you should shade how many parts?
Ls: Thirty three.
T: And in one circle, how many parts should be there?
Ls: Twelve!
T: Okay, and in the last circle, how many parts you should shade?
Ls: Nine.
T: Out of?
Ls: Twelve.
T: Very good. (Appendix I: lines 201-214)

This paragraph presents learners' responses to two open ended questions that aimed at reinforcing the conceptual properties of proper fractions and improper fractions in relation to one whole unit. The first question was: *What are the common properties for all fractions you have written above [in worksheet 3]?* This question aim was to reinforce properties of fractions less than one unit. Four learners shared their answers to the question with the class. L9 explained that *"all fractions are written in the proper fraction form" (Appendix I: line 542)*. L1 indicated that *"all fractions are proper fractions because they have smaller* *numerators and bigger denominators" (Appendix I: lines 544).* L8 said *"all fractions are smaller than one" (Appendix I: line 546).*L6 told the class that *"the fractions are all proper fractions. They are less than one [unit] and the numerators are smaller than the denominators"* (Appendix I: lines 551-552).The response of L6 to the question was regarded as a rich answer. The second question was*: When is it possible to give your answer as improper fractions or mixed numbers?* L6 indicated that she *"learned that proper fractions cannot be converted into mixed numbers" (Appendix I: lines 274).* L7 indicated that it is only *"when you have a whole number and more" (Appendix I: line 427).* The response of L7 to the question was regarded as sound and valid. The teacher reminded the class that it is only possible to identify a fraction as an improper fraction (mixed fraction) if at least one whole unit is shaded, so that the number of whole units shaded represent the whole number of a mixed fraction and the number of shaded parts in the last partly shaded unit represent the proper fraction of a mixed fraction.

5.2.4 Using benchmarking and rules for comparing and ordering fractions

This subsection presents the process of the interaction that enabled learners to use benchmarking and rules for comparing and ordering fractions. Benchmarking is a procedure for comparing fractions by relating their positions to reference points 0, 1/4, 1/2, 3/4 and 1 on the number line (Clarke et al., 2011). The rules for comparing and ordering fractions refer to procedures for comparing and ordering fractions with the same denominator, or fractions with the same numerator.

The use of benchmarking and rules for comparing and ordering fractions were first introduced to the learners during the fifth lesson, and were reinforced continuously from the sixth until the eighth lesson of the intervention. Figure 5.7 shows the cumulative, continuous process of intertwined activities that contributed to the mastery of using benchmarking and rules to compare and order fractions.

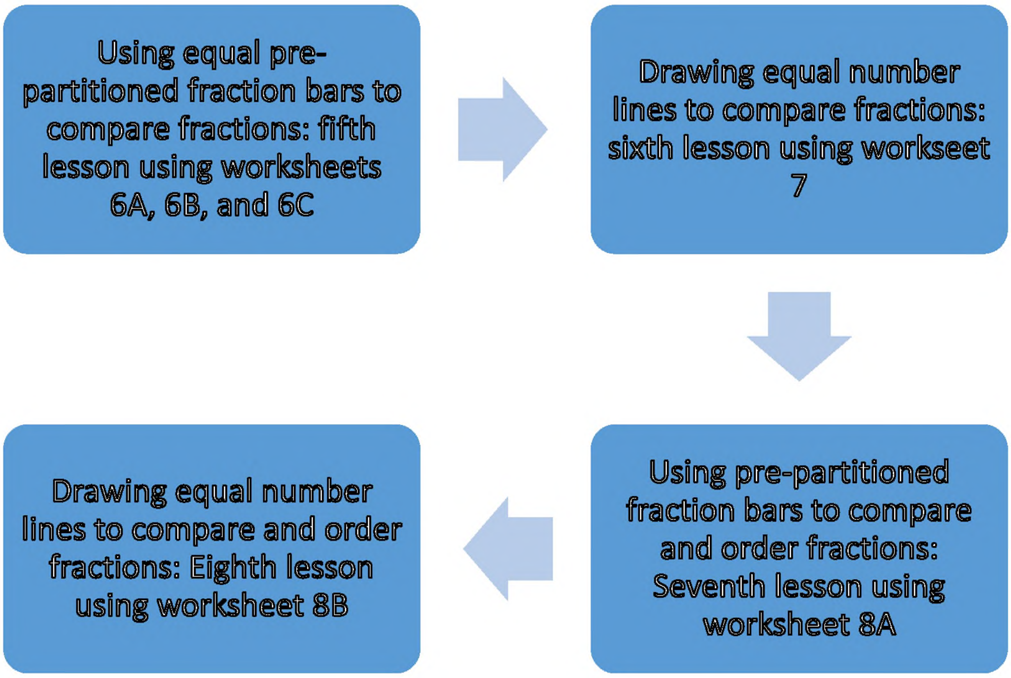


Figure 5.7: The cumulative, linear process for mastering the use of benchmarking and rules for comparing and ordering fractions

The discussion in this subsection focuses on six important issues. Firstly, the subsection presents how learners grappled with shading fraction bars to compare fractions (see extracts 5.18 and 5.19). Secondly, the subsection presents how the teacher helped learners to use fraction bars to compare fractions appropriately (see extracts 5.20 and 5.21). Thirdly, the subsection presents interactions on how the class deduced three rules for comparing fractions (see extracts 5.22-5.25). Fourthly, the subsection presents interactions on how the teacher helped learners to master the use of benchmarking (see extracts 5.26-5.28). Fifthly, the subsection presents snapshots of learners' responses using benchmarking appropriately. Lastly, the subsection presents workings of three learners (L4, L5 and L8) who did not master the use of benchmarking over the period of the intervention.

The fifth lesson made use of worksheets 6A, 6B, and 6C. Each of the worksheets consisted of examples of one type of fractions. Worksheet 6A consisted of fractions with the same numerators; worksheet 6B consisted of fractions with the same denominators, while worksheet 6C consisted of fractions with different numerators and denominators. The grouping of one type of fraction into one worksheet aimed at helping learners to observe and deduce the pattern (rule) for comparing and ordering similar fractions. This pattern was to be deduced by answering the following question in every worksheet: *What did you find special* *about comparing these fractions?* In every worksheet, learners were expected to determine the sizes of fractions they were comparing by shading the appropriate area onto equal pre-partitioned fraction bars.

During the fifth lesson, the teacher started by handing out worksheets 6A, 6B and 6C to the class. Learners were told to make use of shading fraction bars to determine the sizes of fractions they were comparing. When the learners started to complete the worksheets, many compared fractions without shading fraction bars and then they got incorrect answers. They used the sizes of the denominators to compare fractions so that the bigger fraction was the one with the bigger denominator and the smaller fraction was the one with the smaller denominator. For instance, in worksheet 6A, L1, L4, L5, L8, L9, L10, L11 and L12 concluded that 1/4>1/3, 3/4<3/9, 2/7>2/5, 4/5<4/10, 7/10>7/7, and 5/6<5/8. During the one-on-one class interview with L8, he states that he "*thought 1/4>1/3 because 1/4 has the bigger number which is 4 than 1/3*" (Appendix I: line 230). In worksheet 6C, these learners indicated that 1/2<3/8, 2/2<6/6, 2/3<5/10, 6/7>5/6 and 4/5<3/7. As in the pre-test, it was only two learners (L6, L7) that compared fractions in all three worksheets appropriately by using shading of fraction bars and using lowest common denominator method to confirm their answers.

L2 used a different inappropriate procedure to compare fractions in the worksheets. She used the numerators as divisors to compare fractions so that a fraction with a smaller numerator was the bigger fraction and a fraction with a bigger numerator was the smaller fraction. For instance, in worksheet 6A, she indicated that 3/4<3/9, because 9÷3>4÷3. In worksheet 6B, she concluded that 5/8>6/8 and 3/10>7/10, because 8÷5>8÷6 and 10÷3>10÷7 respectively. This error was similar to how she compared fractions in the pre-test.

The next two extracts 5.18 and 5.19 present how the teacher helped learners to make sense of shading fraction bars in order to determine the sizes of fractions and to deduce the rules for comparing fractions. Extract 5.18 presents how the teacher helped L2 to understand that the divisors of the fractions are represented by the fraction denominators and the fraction numerators may represent the number of people sharing.

Extract 5.18: Interaction of the fifth lesson showing how L2 was helped to develop appropriate meanings of numerators and denominators when comparing fractions

L2: Sir?
T: Yes!
L2: I was thinking, for example for this fraction 6/8, these are eight pieces that are shared between six people. And then, the number of shaded parts here are eight.
T: No! The shaded parts are the ones on top [above the fraction bar]. And now?
L2: Ooo! These are eight people that are sharing this one?
T: The number of people, the sharers are down [below the division line]. The quantities are on top. …
L2: Ooo!
T: You see. So what were you thinking?
L2: Sir, I was thinking that, I thought 5/8>6/8 because 8÷5 is bigger than 8÷6.
T: Okay, shade now, show five out of eight and shade six out of eight. [L2 shaded the fraction bars to show 5/8 and 6/8, and the teacher probe again].
T: Which one is bigger?
L2: This one (6/8).
T: Yes! And then now?
L2: Ooo!
T: You look at the amount shaded.
L2: Yes sir.
T: *Iyaa* [like that]. (Appendix I: lines 643-658)

Before the end of the fifth lesson, L2 started to compare fractions appropriately by using the area shaded of fraction bars. Extract 5.19 presents the interaction of L2 with the teacher.

Extract 5. 19 Interaction of the fifth lesson showing L2 comparing fractions appropriately by using the shading of equal fraction bars

T: How did know that 2/5>2/7.
L2: First, I looked at the area shaded. The shaded part for 2/5 looks bigger than for 2/7.
T: Okay. (Appendix I: lines 731-732)

The next extract 5.20 presents how the teacher convinced learners to use fraction bars to compare fractions appropriately.

Extract 5. 20 Interaction of the fifth lesson showing L8 learning to use fraction bars to compare fractions as expected

L8: 1/4>1/3
T: … But, did you use these fraction bars to get your answers?
L8: No, sir!
T: Can you shade these fraction bars for me? ... Shade for me one fourth and one third here. Shade. That's what?

L8: Ooo!
T: *Owu wete* [you see]? *Iyaa* [like that]. Check your answers. I put these things [fraction bars] for you to check your answers. Which one is bigger?
L8: This one (1/3).
T: Now, good, do the same with others. (Appendix I: lines 674-680)

Extract 5.21 presents how L12 concluded that 5/6<5/8 without shading the fraction bars and how the teacher helped him to use fraction bars to compare fractions appropriately.

Extract 5. 21 Interaction of the fifth lesson showing L12 learning to use equal fraction bars to compare fractions as expected

L12: 5/6<5/8.
T: Did you use the fraction bars.
L12: No!
T: Mhh? What did you shade there, how did you shade if you did not? I want you to look at the fraction bars and try to make some sense.
L12: Yes, sir.
T: Where is this fraction here [which fraction bars is for which fraction]? …
L12: Is this one.
T: Okay. Which one is having more shaded area?
L12: Is this one (5/6).
T: And then, you did not even realise. You see. It is like that. These [fraction bars] are here to help you to see that 5/6>5/8, because [when you] look at how the eighth is, this area [shaded] is very smaller comparing to the other one [of the sixths].
L12: Yes, sir.
T: That's what I wanted you to understand.
L12: Okay, sir. (Appendix I: lines 711-730)

The next three extracts 5.22-5.24 present how the teacher helped learners to deduce the three rules for comparing fractions in worksheets 6A, 6B and 6C by interpreting and answering this question: *What did you find special about comparing these fractions?* Extract 5.22 present the interaction on deducing the rule for comparing fractions with the same numerators.

Extract 5. 22: Interaction of the fifth lesson showing L7 deducing the rule for comparing fractions with the same numerator

T: I am going to help you to answer this question [what did you find special about comparing these fractions?]. Those fractions have a common property and when you look at the way you compared those in worksheet 6A, what is so special here? The numerators are what? [L8 and L10 state that the numerators were the same].
T: Now, if someone tell you, all these fractions look the same, I want you to tell me how will you know which one is greater?
L10: You look at the shaded parts.
T: Yes, and shaded parts are?
L7: There in [worksheet] 6A, the numerators are the same, so you look at the denominators. The smaller the denominator, the bigger the value of the fraction. (Appendix I: lines 737-751)

The phrase by L7 in extract 5.22 was adopted as the rule for comparing fractions with same numerators in worksheet 6A and regarded as the appropriate answer to the open ended question in worksheet 6A. The teacher wrote this rule on the chalkboard this way: *If the numerators are the same, then the smaller the denominator, the bigger the fraction*. The teacher drew the attention of the class to the fractions in worksheet 6A on the chalkboard. The teacher and the class used the rule for comparing fractions with the same numerators to conclude that 1/4<1/3, 3/4>3/9, 2/7<2/5, 4/5>4/10, 7/10<7/7 and 5/6>5/8. The teacher also informed the class that the inequality sign should always face the fraction with the smaller denominator when comparing pairs of fractions with the same numerators. Extract 5.23 presents the interaction on how the teacher and class used the rule of comparing fractions with same numerators to compare fractions in worksheet 6A.

Extract 5. 23: Interaction of the fifth lesson showing the teacher and class using the rule to compare fractions with the same numerator

T: Like here [2/7<2/5], five is smaller than seven, *ne*?
Ls: Yes.
T: Here [3/4>3/9], four is smaller than nine?
Ls: Yes.
T: Here [1/3>1/4], three is smaller than four?
Ls: Yes.
T: But the numerators for the first one [first pair of fractions] are the same. For the second one are the same, for the third one are the same and for the fourth one are the same.
Ls: Yes.
T: So, we conclude that, if the numerators are the same, then the smaller the denominator, the bigger the fraction. We have seen how we used fraction bars to compare fractions, *ne*?
Ls: Yes.
T: Okay. This is exactly what I wanted you to write when you were to answer this question.
Ls: Yes, sir.
T: This rule will save you energy and time. Write it down and never forget it, *ne*?

Ls: Yes, sir. (Appendix I: lines 772-793)

Extract 5.24 presents the interaction on deducing the rule for comparing fractions with the same denominators.

Extract 5. 24: Interaction of the fifth lesson on deducing the rule for comparing fractions with the same denominators

T: Let's move to worksheet 6B. This is almost the same as fractions in worksheet 6A, the denominators [in here] are the same.
T: Which one is bigger here in a?
Ls: Two over five.
T: Here?
Ls: Six eighths. …
T: The sign should face the bigger fraction, *ne*?
Ls: Yes, sir.
T: Now, what is so special about comparing these fractions?
L10: If the denominators are the same, the bigger the numerator, the bigger the fraction or value.
T: Yes, thank you very much. (Appendix I: lines 798-809)

The phrase by L10 in extract 5.24 was adopted as the rule for comparing fractions with the same denominators. The teacher wrote this rule on the chalkboard this way: *If the denominators are the same, then the bigger the numerator, the bigger the fraction or value*. Extract 2.25 presents interaction on deducing the rule for comparing fractions with different numerators and denominators.

Extract 5. 25: Interaction of the fifth lesson on deducing the rule for comparing fractions with different numerators and denominators

T: Let's move to worksheet 6C, *ne*?
Ls: Yes, sir.
T: What can you say about comparing these fractions [in worksheet 6C]?
L7: Check which fraction is closer to a half or to a whole.
T: Mhh! Now, when you look at the numerators and denominators of these fractions, they are all different. You need to compare which fraction is closer to zero, which one is closer to a half, which one is closer to three quarters and which one is closer to one.(Appendix I: lines 832-836).

The phrase by L7 was adopted as the use of benchmarking to compare fractions. The teacher wrote this rule on the chalkboard this way: when the numerators and denominators are different, then check how a fraction is closer to 0, 1/4, 1/2, 3/4, or 1.

The next three extracts 5.26-5.28 present how the learners made use of benchmarking to compare and order fractions during the sixth lesson up to the eighth lesson using equal number lines and fraction bars from worksheets 7, 8A and 8B.

During the sixth lesson, learners compared fractions by drawing equal number lines. The teacher started the lesson by demonstrating to the class how to use different fraction denominators to partition the number lines with equal number of units.

Extract 5. 26 Interaction of the sixth lesson showing the teacher explaining how to compare fractions using equal number lines

T: You know what I advise you to do first when you are drawing the number lines, first mark the position of zero on both number lines. Then decide where to put one first on both number lines. They [whole numbers] should be straight. You need to keep their length always equal. Otherwise, you will get it wrong, *ne*?
Ls: Yes, sir.
T: Then you can start to make equal parts [unit subdivisions] using a ruler, *ne*?
Ls: Yes, sir.
T: The only difference should be how many parts are here [on this number line] and how many parts are here [on the second number line]. And, you should always partition the number lines using the fraction denominators.
Ls: Yes, sir. (Appendix I: lines 1004-1021)

After demonstrating these techniques on the chalkboard, learners started to complete worksheet 7. As they compared fractions, learners were expected to write a brief statement explaining how they applied benchmarking to obtain their answers. Extract 5.27 presents the interaction of the teacher explaining to the class that they should write a brief statement for every pair of fractions they were comparing.

Extract 5. 27 Interaction of the sixth lesson showing a teacher explaining to the class that they need to write a brief supporting statement for their answers

T: Yes. I want you to give a short sentence to explain how you knew that this … [fraction] is bigger than the other fraction. I want you to use two number lines to reason.
Ls: Yes, sir. (Appendix I: lines 959-963)

It took lots of explanation to help learners to make sense of benchmarking to compare fractions. Extract 5.28 presents how L4 used inappropriate reference points when he compared 4/5 and 2/3.

Extract 5. 28: Interaction of the sixth lesson showing L4 grappling with the use of benchmarking

T: 2/3 is closer to what?
L4: One.
T: This [4/5] is the one which is closer to one. This (2/3) is closer to what? There are many numbers here [on the number line], you can use zero, a half, or use three quarters.
L4: Mhh!
T: Where is a half here?
L4: Here.
T: That's one. Where is a half of one?
L4: Is here.
T: No … where do you think a half is from zero to one?
L4: Is here.
T: Is this where a half is? No, show me where the middle of this line is? A half is here, you see.
L4: Yes, sir.
T: It is always there even if it is not shown (Appendix I: lines 986-1001).

Figure 5. 8 presents snapshots of learners' responses for using benchmarking to compare and order fractions appropriately, which were extracted from worksheets 7, 8A and 8B.

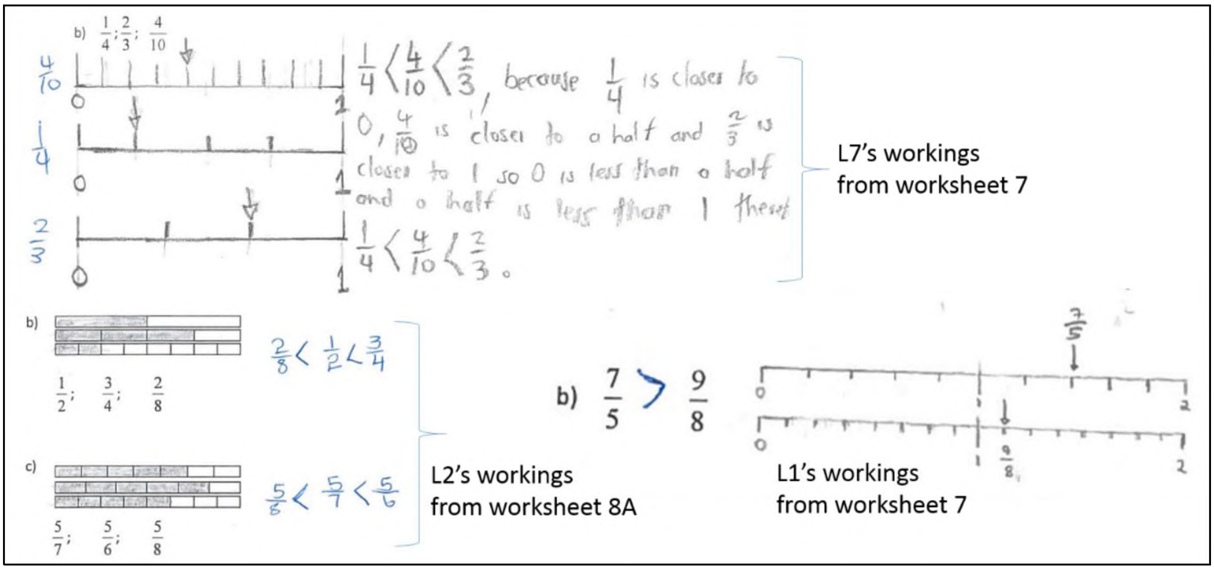


Figure 5.8: Workings of L1, L2 and L7 extracted from worksheets 7 and 8A of the seventh and eighth lesson showing the use of benchmarking, number lines and fraction bars for comparing and ordering fractions

Three learners – L4, L5, and L8, showed unstable understanding of benchmarking. Instead of relating the sizes of fractions to the shading of fraction bars and number lines, they compared and ordered fractions using the sizes of fraction denominators. Figure 5.9 shows a snapshot of their responses extracted from worksheets 8A and 8B.

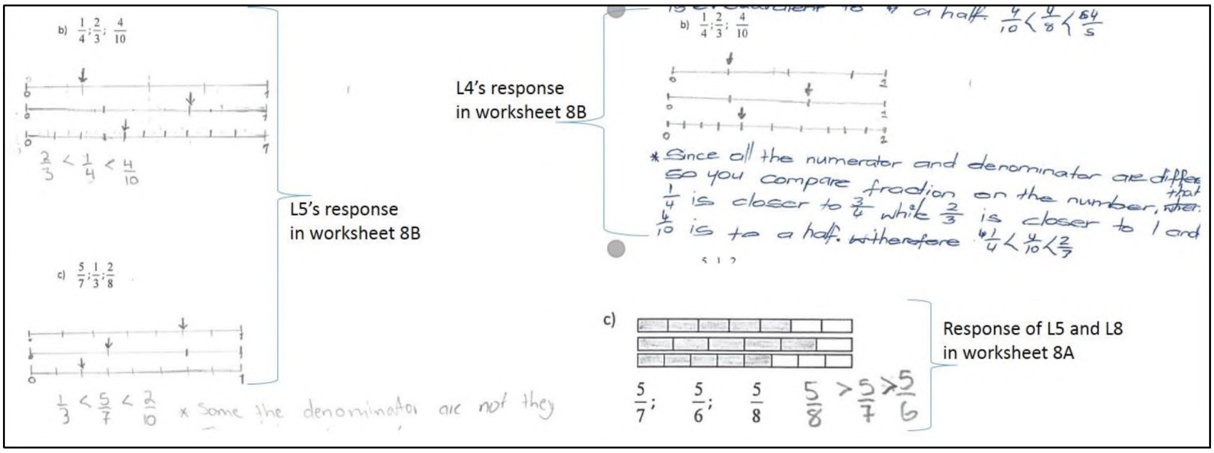


Figure 5.9: Workings of L4, L5 and L8 showing the use of denominators for comparing and ordering fractions during the seventh and eighth lessons

The snapshot above presents one of the major difficulties that L4, L5 and L8 showed at the end of the intervention, particularly in the post-test. These learners compared fractions using the sizes of denominators as they did not relate the sizes of fractions to their positions on the number lines as it was expected.

5.2.5 Visually representing the lowest common denominator method for adding two fractions

This subsection presents the process of interaction that influenced learners' conceptual understanding of the lowest common denominator method for adding fractions. The discussion of this subsection presents three important issues. Firstly, the subsection presents the interaction on how the teacher and class worked out the sum of fractions using the lowest common denominator method (see extract 5.29-5.31). Secondly, the subsection presents interaction on how the teacher helped learners to use fraction bars to visually represent the lowest common denominator (LCD) method for adding two fractions (see Figure 5.10). Thirdly, the subsection presents learners' responses showing inappropriate and appropriate use of fraction bars to visually represent the LCD method of adding fractions (see Figures 5.11 and 5.12; extracts 5.32 and 5.33).

The teaching of addition of fractions took place in one lesson – the ninth lesson, using one worksheet only: worksheet 9. It was expected of the learners to use equal fraction bars to graphically illustrate the process of finding the sum of two fractions. Extract 5.29 presents the first interaction for the introduction of the ninth lesson.

Extract 5. 29: Interaction showing the introduction of the ninth lesson

T: We have learned three methods of showing our understanding of fractions as numbers by drawing a circle or using fraction bars or using a number line, *ne*?
Ls: Yes.
T: Now, when we add fractions, we should always check whether the denominators are the same, *ne*? Ls: Yes, sir.
T: If they are not the same, what should we do?
Ls: We look for the lowest common denominator (LCD).
T: Okay, now, without wasting much time, I want us to understand where is this LCD coming from and I want us to do so using a picture to make sense of why it is important to only add fractions once we have the LCD, *ne*?
Ls: Yes, sir.
T: In the first lesson, we learned that we only count parts if they are of equal size, *ne*?
Ls: Yes.
T: Okay. Let's have two examples [1/2+1/2= and ¼+2/5=] to demonstrate this.
Ls: Yes, sir. (Appendix I: lines 1154-1170)

The next extract 5.30 presents the interaction for finding the sum of fractions with the same denominator using LCD method: 1/2+ 1/2.

Extract 5. 30: Interaction of the ninth lesson showing addition of fractions with the same denominator using LCD method

T: Now, we are adding fractions. The first thing is to check if the denominators are the same. Are they the same?
Ls: Yes.
T: Yes, then we can add the numerators and keep the denominators the same, *ne*?
Ls: Yes. (Appendix I: lines 1171-1175)

The teacher and the learners worked out the addition of 1/2 + 1/2 on the chalkboard which simplifies to 1. Extract 5.31 presents the interaction for finding the sum of fractions with different denominators using LCD method: 1/4+2/5.

Extract 5. 31: Interaction of the ninth lesson showing addition of fractions with different denominators using LCD method

T: Here we are given 1/4 + 2/5, *ne*?
Ls: Mhh!
T: Now, the denominators are different, we cannot add. What should we do to make them the same?
Ls: By looking for the LCD of four and five.
T: *Ne*?
Ls: Yes.
T: Which is?
Ls: Twenty.
T: Twenty. And how do we make four to become twenty?
Ls: By multiplying with five.
T: So, then you will have… (Appendix I: lines 1182-1191)

Thereafter, the teacher and class used cross multiplication of denominators and BODMAS on the chalkboard to show that 1/4 + 2/5= 1/4 × 5/5 + 2/5 × 4/4 = 5/20 + 8/20 = (5+8)/20 = 13/20. After this demonstration, teacher introduced the use of fraction bars to visually represent the conceptual meaning of the LCD method. Figure 5.10 presents a snapshot of how the teacher demonstrated the use of fraction bars for addition of fractions with same denominator by proving that a half (1/2) of a fraction bar shaded plus a half (1/2) of a fraction bar shaded add up to one fully shaded fraction bar for 1/2 + 1/2.

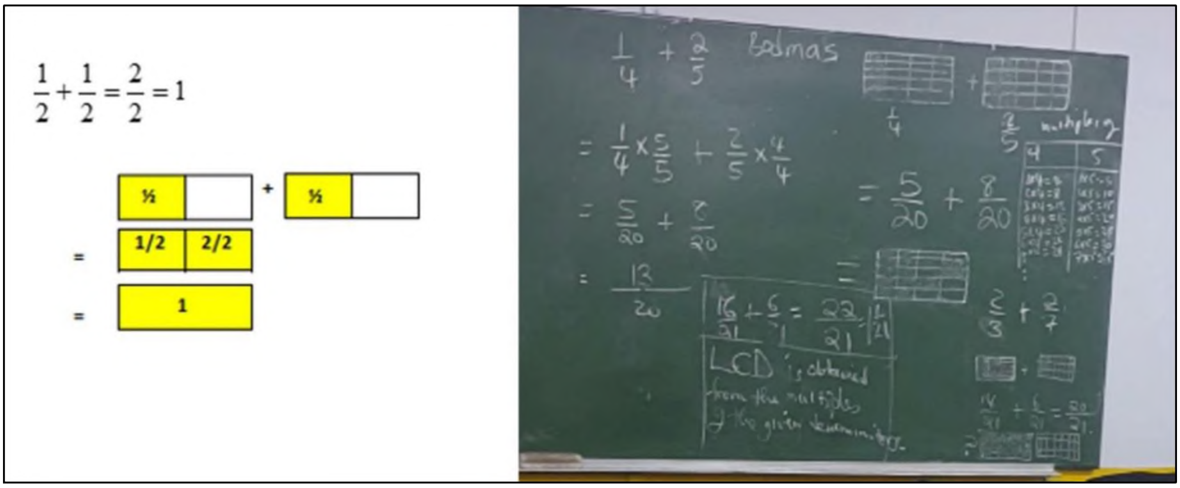


Figure 5.10: Workings from the chalkboard during the ninth lesson showing the use of fraction bars to graphically represent the lowest common denominator method for adding fractions

The next extract 5.32 presents how the teacher demonstrated to the class using an example of two fractions 1/4 + 2/5, the use of fraction bars to visually represent the conceptual meanings of finding the LCD when adding fractions with different denominators.

Extract 5. 32: Interaction of the introduction during the ninth lesson on using fraction bars to visually represent the use of LCD when adding fractions with different denominators

T: Do you think it really make sense for us to find the LCD?
Ls: Yes.
T: Let's show. Now let's show that using the picture.
T: To make these parts equal… use this five to partition horizontally the first fraction bar and also use this four to partition horizontally the second fraction bar, *ne*?
Ls: Yes, sir. (Appendix I: lines 589-591)

The partitioning of the first fraction bar and the second fraction bar presented in extract 5.32 implies that each fraction bar would have five parts shaded out of twenty equal parts and four parts shaded out of twenty equal parts, respectively. Figure 5.13 presents the snapshot of the chalkboard showing the use of fraction bars to visually prove 1/4 + 2/5 = 13/20.

Learners were amazed to see the graphical representation of the sum of fractions 1/4 + 2/5 on the fraction bars and they did not hold their excitement as many started to shout loudly: wow, wow, awe, awe, wow! The teacher also explained to the class that, learners need to know the minimum number of fraction bars needed to represent the sum of fractions. For instance, in case of 13/20, only one fraction bar was needed to show 13/20 since the number of shaded parts (13) were less than the number of parts that make up the whole fraction bar. Finally, the teacher informed the class that the principle for finding the LCD to add fractions with different denominators resembles the rule of counting, that is only equally sized units can be counted together; and that the equally sized units are obtained by repartitioning the fraction bars as shown in Figure 5.10.

Thereafter, learners were asked to complete worksheet 9 individually, first by calculating the sum before visually representing the sum using fraction bars.

One of the challenges in this lesson was some learners who did not follow instructions as they chose to partition the fraction bars before calculating the sum and this choice made these learners not to learn much from this lesson as their conceptual representation of the sum of fractions with different denominators did not improve. The next extract 5.33 presents the interactions on how the teacher helped the class to use fraction bars to represent the conceptual meaning of the LCD using 1/6+2/3 as an example.

Extract 5. 33: Interaction of the ninth lesson showing interaction of the teacher helping learners to use the fraction bars to represent the conceptual meaning of the LCD

T: You did not show your calculations, why? First, find the sum before you start using the fraction bars, *ne*?
L4: Okay, sir.
T: But, your parts are not the same, make them equal in both fraction bars.
L4: Yes, sir. … [L7 used the LCD method to obtain the correct sum 5/6].
T: Explain now?
L7: Six becomes the LCD of three and six.
T: Okay.
L7: Then, I divided all of these fraction bars into six parts. I have already shaded two thirds which becomes four sixths, and this one-sixth still remains the same, because the parts were already six. Now, I added four plus one then I get five out of six.
T: Okay (Appendix I: lines 1365-1386).

Figures 5.11 and 5.12 show snapshots of inappropriate and appropriate learners' workings respectively, by using fraction bars to represent addition of two fractions with different denominators: 2/3 + 1/6 and 1/2 + 1/3.

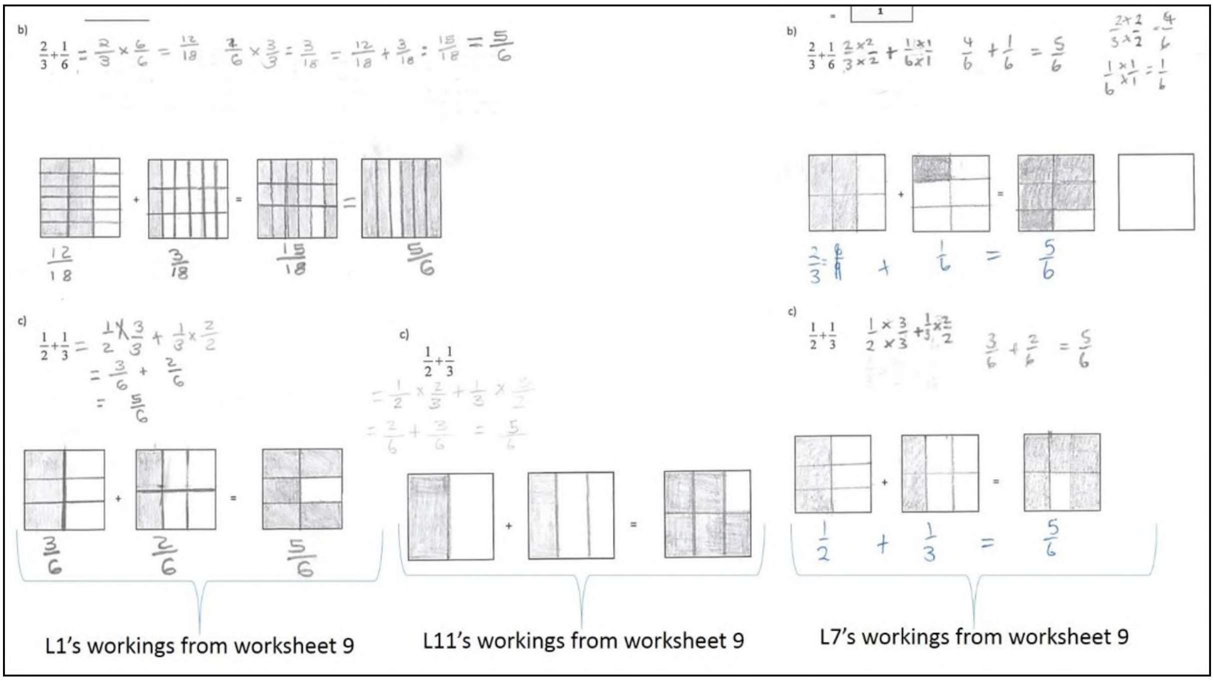


Figure 5.11: Workings of learners extracted from worksheet 9 showing the appropriate use of fraction bars to visually represent the lowest common denominator method for adding two fractions

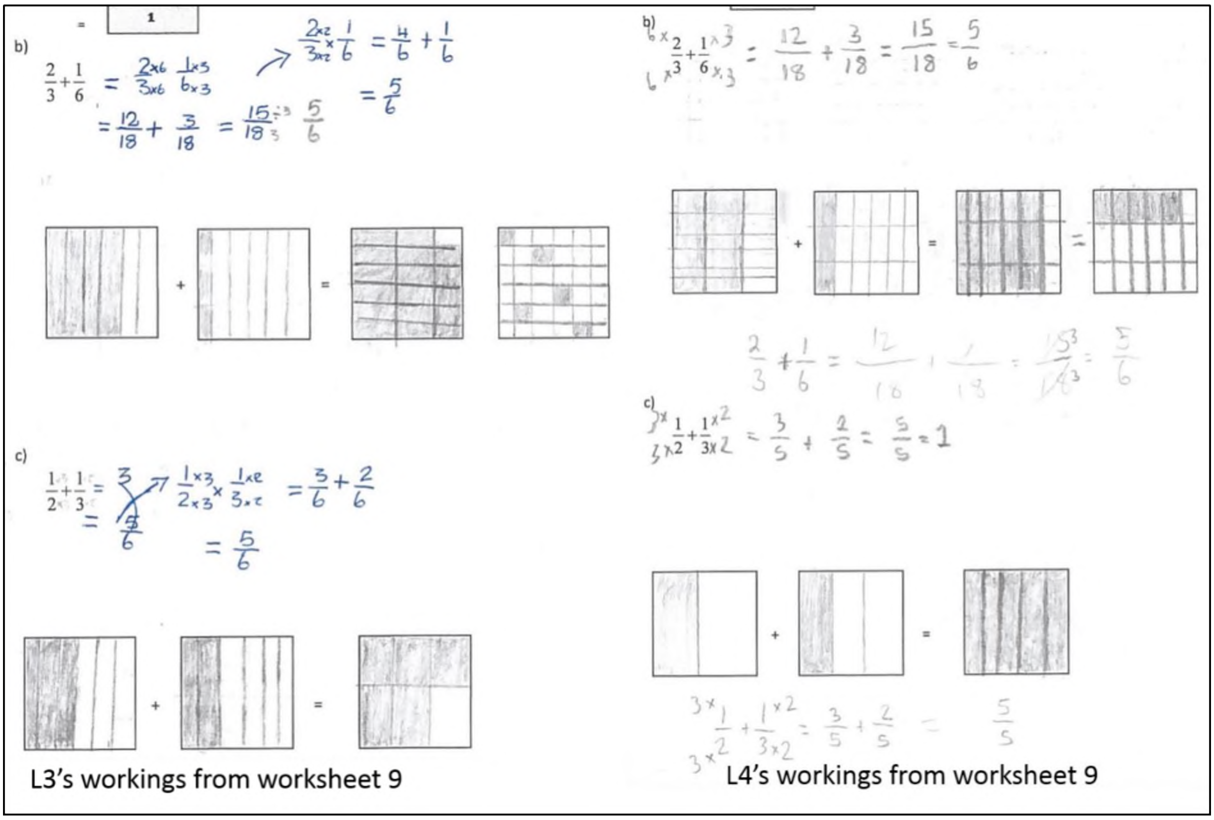


Figure 5. 12: Workings of learners extracted from worksheet 9 showing inappropriate use of fraction bars to visually represent the lowest common denominator method for adding two fractions

5.3 CONCLUSION

This section presented five factors on the influence of the teaching intervention. The first three factors focused on the process for developing a conceptual understanding of fractions. This process involved five worksheets which were used in the first four lessons of the intervention. The last two factors presented, focused on the process for developing conceptual understanding of and procedural fluency with fractions. This process involved eight worksheets which were used during the last five lessons of the intervention.

CHAPTER SIX

DISCUSSION OF RESEARCH FINDINGS

6.1 INTRODUCTION

This chapter presents the discussion of research findings based on the data analysed in Chapter Four and Five, and the literature review in Chapter Two. The discussion of the research findings is organised into four sections, namely: sections 6.2 - 6.5. Section 6.2 presents a brief overview of the themes identified in relation to each research question and shows the links of the identified themes to the data presented in Chapter Four and Five. Sections 6.3 - 6.5 presents a detailed discussion of these themes in relation to the literature review in Chapter Two. The findings of this research are all presented in the form of analytical statements related to the research questions.

6.2 OVERVIEW OF THEMES IDENTIFIED

The purpose of the present study was to investigate the use of two fraction models to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions. To do this, it sought to answer the following research questions:

- What was the nature of Grade 8 learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention?
- How did Grade 8 learners' conceptual understanding of and procedural fluency with fractions change during the teaching intervention?
- How did the teaching intervention influence Grade 8 learners' conceptual understanding of and procedural fluency with fractions?

The findings related to each research question can be organised thematically. This section briefly describes each theme. With respect to the first research question, analysis of the pre-intervention results suggested that these Grade 8 learners displayed a number of difficulties in their conceptual and procedural engagement with fraction models and fraction symbols. Four major difficulties were identified as themes:

- These learners read fractions using inappropriate names (see paragraph 6, subsection 4.3.1.1).

- These learners did not identify the whole unit in the models and therefore identified fractions represented by the fraction models, using inappropriate fraction symbols (see subsection 4.3.1.1).
- These learners compared and ordered fractions inappropriately using the sizes of the numerators and denominators separately. This theme relates to the inappropriate procedures used for comparing and ordering fractions of subsection 4.3.2.1 in Chapter Four.
- These learners used the lowest common denominator method inappropriately for adding fractions with different denominators. This relates to the inappropriate procedures used for adding fractions of subsection 4.3.3.1 in Chapter Four.

These themes are discussed in detail in relation to the literature, in section 6.3.

A number of changes in learners' conceptual understanding of and procedural fluency with fractions (question 2) were identified over the time period of the intervention. Four particular areas of conceptual and procedural development were identified as themes. The themes are:

- Learners were now able to identify fractions represented using area models and number lines and to describe these fractions using appropriate names (see paragraph 3 of subsection 4.4.1.1).
- Learners were now able to identify the whole unit in area models and number lines and to develop a sense of the size of fractions in relation to one whole unit (see paragraph 4 of subsection 4.4.1.1).
- Learners were now able to conceptually use equal fraction bars, equal number lines, benchmarking and rules for comparing and ordering fractions. This theme relates to the appropriate procedures for comparing and ordering fractions shown in subsection 4.4.2.1 in Chapter Four.
- Learners became able to use equal fraction bars to visually represent the lowest common denominator method for adding fractions and to recognise that only equally sized units can be counted together. This theme relates to the appropriate procedures used for adding two fractions in subsection 4.4.3.1 of Chapter Four.

These themes are discussed in detail in relation to the literature review in section 6.4.

For the third research question, four key themes relating to the possible influence of the teaching intervention on the changes in learners' conceptual understanding of and procedural fluency with fractions, were identified. These themes are:

- Using area models and number lines to identify both fraction symbols and appropriate fraction names of areas shaded, seemed to help these learners to see fractions as relational numbers. This theme relates to reading fractions using appropriate names in subsection 5.2.1.
- Prompting to partition whole units of the fraction models and graphically illustrating fraction symbols, seemed to help these learners to identify the whole unit in fraction models and to develop a sense of the size of fractions in relation to one whole unit. This theme relates to the results presented in subsections 5.2.2 and 5.2.3 of Chapter Five.
- Using the area model and number lines to graphically illustrate fraction symbols seemed to help these learners to use equal fraction bars, equal number lines, benchmarking and rules for comparing and ordering fractions. This theme relates to the results presented in subsection 5.2.4 of Chapter Five.
- Using equal fraction bars to graphically illustrate fraction denominations seemed to help these learners to visually represent the lowest common denominator method for adding fractions and to recognise that only equally sized units can be counted together. This theme relates to the results presented in subsection 5.2.5 of Chapter Five.

These themes are discussed in detail in relation to the literature review in section 6.5.

6.3 LEARNERS' PROFICIENCY WITH FRACTIONS BEFORE THE TEACHING INTERVENTION

This section presents and discusses the research findings relating to the nature of the learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention. The analysis of learners' responses suggested four major difficulties in their conceptual and procedural engagement with fraction models and fraction symbols. The findings are presented in two subsections, namely: conceptual understanding of and procedural fluency with fractions. Although the findings are presented in two subsections, the nature of procedural difficulties displayed by the learners suggested that these difficulties were linked to conceptual misunderstandings. Two themes were identified for each subsection.

6.3.1 Conceptual understanding of fractions

This subsection presents two themes relating to the nature of conceptual understanding of fractions, shown by the participants of this study. The analysis of learners' responses in the pre-test and pre-interview clearly indicated that the majority of the learners who participated in this research had difficulties relating unit subdivisions in the models appropriately to the denominator of fraction symbols (see subsection 4.3.1.1 of Chapter Four). Two themes could be identified in this regard. Firstly, the learners read fractions using inappropriate names. Secondly, the learners did not identify the whole unit in the models and therefore identified fractions represented by the fraction models using inappropriate fraction symbols.

6.3.1.1 Reading fractions using inappropriate names

The analysis of the pre-intervention results showed that the learners tended to read fractions using inappropriate names. They did this in two ways. The first form of reading fractions inappropriately was to use the word *"quarters"* to name fractions that were not quarters. For instance, in the pre-test, some learners described the fraction 5/12 either as *"five quarters"* or *"five quarters of twelve"* (see paragraph 6 in subsection 4.3.1.1 of Chapter Four). These two names were used to refer to 5/12 both as the fraction represented by a shaded area of the circle and as a fraction represented symbolically. The second form of reading fractions inappropriately is by reading fractions as two unrelated whole numbers *"over"* one another. For instance, in the pre-test, they referred to 5/12 as *"five over twelve"*.

The reading of fractions using the word *"over"* suggested that these learners appeared to think of fractions as a pair of two separate unrelated whole numbers as discussed by Clarke et al (2011). Clarke et al (2011) state that reading fractions as two whole numbers over one another is confusing and makes it difficult for the learners to tell which of the digits of the fraction symbol represent the number of parts or the size of the parts, in the unit. In the present study, the learners displayed this difficulty representing the number of parts in the unit by using the word *"quarters"* to name fractions which are not quarters. Reading of fractions using inappropriate names could also be related to the way fraction concepts are taught in the Namibian curriculum. For instance, in the Namibian fifth grade syllabus

(Namibia. Ministry of Education, 2010), learners are expected to identify the numerator and denominator using the given fraction notation, but the teaching of appropriate fraction names is not included. According to my own classroom teaching experience, fractions are only read as two whole numbers over one another. This use of inappropriate names to read fractions, agrees with the findings of Vatilifa (2012). According to Vatilifa (2012), the teaching of fractions in Namibia is currently taught only through symbolic representations of abstract fraction concepts, and an incorrect use of "terminologies such as '1 over 4' or '1 out of 4' instead of a 'quarter' or 'one-fourth'" is common (Vatilifa, 2012, p. iii). Her study also reveals that some student teachers interpret a fraction as a pair of two different whole numbers that can be broken apart (Vatilifa, 2012, p. iii). Adapting the Grade 5 curriculum to include the teaching of appropriate fraction names, may help to address this issue and enable learners to think of fractions as relational numbers.

6.3.1.2 Not identifying the whole unit in models and therefore identifying fractions using inappropriate fraction symbols

Wiest et al. (2015) stipulate that conceptual understanding of fractions involves understanding the meaning of the fraction numerator and denominator, as well as their relationship to each other in a holistic manner. In this study, the analysis of the pre-intervention results showed that the majority (11 of the 12 learners), appropriately related the fraction numerator and denominator to the shaded parts and the total number of equal pieces, in a single area model, using correct fraction symbols (see subsection 4.3.1.2). But, of these 11 learners, only two used appropriate fraction symbols to represent fractions in area models and number lines showing more than one unit (see paragraph 2 of subsection 4.3.1.2).

The analysis of the pre-intervention results revealed that most of the learners had a number of difficulties identifying units in area models, with multiple wholes and on the number line. In the first instance, all learners showed difficulties in locating a proper fraction on a number line showing more than one unit. For instance, many learners located 3/4 between 3 and 4 (see paragraph 4 of subsection 4.3.1.1). The locating of fractions less than one unit, after one, is identified by Mitchell and Horne (2011) as the most rampant misconception shown by learners with a limited part-whole interpretation of fractions, when solving number line tasks. They indicate that many learners commit this error by regarding the length of the number line as a simple line whose full length represents a whole. Larson (1987) agrees with Mitchell and

Horne (2011) that many learners make the error of reading fractions on the number line by disregarding the scaling and treating the whole line as a unit.

In the second instance, the majority of learners showed difficulties in identifying the unit in part-whole diagrams and number lines showing more than one unit and used inappropriate fraction symbols to represent these fractions (see paragraphs 7-11 of subsection 4.3.1.1).In the third instance, some learners used decimal notation, whole numbers and inappropriate fraction notation such as *1/ (number of shaded parts)* and *(number of shaded parts)/1* to identify fractions on a number line, which was one unit long (see paragraphs 2-3 of subsection 4.3.1.1). Mitchell and Horne (2011) and Van de Walle et al. (2013), identify the use of incorrect fraction notation such as a decimal notation and whole numbers, as some of the common errors that learners working with a fraction number line, do make.

The conceptual understanding of fractions demonstrated by the learners of this study resembles the findings of a survey of 3067, Finnish 5th and 7th graders, whose thinking appeared to be dominated by the part-whole interpretation of fractions (Hannula, 2003). These students showed difficulties in perceiving a proper fraction as a number on the number line, though they could find the same proper fraction of a single fraction bar. According to Kerslake (1986), his research with 12-14 year old students (whose age group is closer to the age group of the learners in this study: 13-16 years old) suggests that the use of part-whole models alone, does not prompt learners to think of fractions as numbers, but only as shaded parts over a total number of pieces in the whole unit. Clarke, Roche and Mitchell (2011) also describe the part-whole interpretation of fractions as an insufficient foundation to develop a good conceptual undersanding of fractions.

The limited part-whole interpretation of fractions demonstrated by the learners of this research study, may have arisen from the form of fraction instruction offered in the Namibian mathematics classroom. The analysis of Namibia's mathematics textbooks for Grade 5 and 6 presented in section 2.2 of Chapter Two, showed that the part-whole interpretation of fractions is only related to a single whole area drawn as a circle or rectangle, or in a fraction chart, to identify fractions whose numerator is smaller than the denominator.

Common challenges for the learners in this study, were to identify the unit and to relate the unit subdivisions to the denominator of the fractions. Amato (2005) believes that learners whose experience of fractions has been limited to part-whole interpretation of a single whole, generally find it difficult even to identify the unit in part-whole diagrams showing more than one unit and this was true for the majority of the learners in this study. The findings presented above agree with the findings by Larson (1987) that many learners have difficulties in identifying the unit on the number line and relating the number of divisions in each unit, to the denominator of fractions, when solving the number line tasks.

6.3.2 Procedural fluency with fractions

This subsection focuses on the types of procedures used for comparing and ordering fractions and adding fractions, as well as on how these procedures were applied to solve fraction tasks of comparing, ordering and adding fractions. The analysis of the pre-intervention results identified two themes, both involving the inappropriate use of procedures to compare, order and add fractions. The two themes presented are related to the analysis of data presented in subsections 4.3.2.1 and 4.3.3.1 of Chapter Four. The first theme is that Grade 8 learners compared and ordered fractions inappropriately, using the sizes of the numerators and denominators separately. The second theme is that Grade 8 learners used the lowest common denominator method inappropriately, to add fractions with different denominators. The nature of the procedural difficulties, suggested that these are related to the conceptual misunderstandings presented in subsection 6.3.1. These two themes are discussed in detail below.

6.3.2.1 Using the sizes of the numerators and denominators separately

The analysis of pre-intervention results showed that some of the learners in this study used some appropriate procedures to compare and order fractions in the pre-test, namely: converting the given fractions to fractions with a (lowest) common denominator; using unequal fraction bars to determine the sizes of the given fractions; and using the rules for comparing fractions with the same numerators and fractions with the same denominators (see subsection 4.3.2.2 of Chapter Four).

The analysis of the pre-intervention results also revealed that the majority of the learners had difficulties interpreting fractions as relational numbers and they used at least one of the following two inappropriate procedures to compare and order fractions in the pre-test (see

Table 4.2 in Chapter Four). The first inappropriate procedure involved comparing and ordering fractions using the sizes of denominators only. This procedure was applied in two ways (see part 4.3.2.1.1 of Chapter Four). The first method used the sizes of denominators to compare fractions so that the bigger fraction was the one with the bigger denominator and the smaller fraction was the one with the smaller denominator. For instance, four learners used this method in the pre-test to conclude that 4/5<4/6; 1/2<3/8; 4/12>1/3; 1/2<1/3<1/4; and 2/3<5/5<1/12. The second method used the sizes of denominators to compare fractions so that the smaller fraction was the one with a bigger denominator and the bigger fraction was the one with a smaller denominator. Two of the learners used this second method in the pre-test to conclude that 4/12<1/3 and 1/12<5/5<2/3.

The other procedure used to compare and order fractions inappropriately in the pre-test and pre-interviews, involved using the numerators as divisors (see part 4.3.2.1.2 of Chapter Four). This procedure was used by four learners to conclude that 2/7>3/7; 4/5<4/6; 4/12<1/3; 8/10<5/10<3/10; and 5/5<2/3<1/12 respectively. In the recall interview, L2 explained that he thought 2/7>3/7, because *"first, I thought that the fractions with smallest numerators are always the bigger ones"* (Appendix H: lines 171). In the recall interview, L9 also explained that he thought 2/7>3/7 and 8/10<5/10<3/10, because *"I thought when the denominators are the same, the smaller the numerator the bigger the fraction. ... I thought... ten people who are going to divide in three parts [three tenths] are the one who are going to get the biggest, and this one [eight tenths] is the smaller one"* (Appendix H: lines 196-198 and 202-205).

The use of procedures to compare and order fractions based on the sizes of the numerators and denominators alone is not new. A research study by Stafylidou and Vosniadou (2004) tested 200 students ranging in age from 10 to 16 years, on comparing fractions. Their research results established that students ordered fractions either on the basis of the size of the numerator only or of the denominator only. Those students that ordered fractions on the basis of the numerator, appeared to ignore the denominators and ordered fractions so that as the numerator of a fraction increased, the fraction itself also increased. Other students that ordered fractions on the basis of the denominator ignored the numerators and ordered fractions so that as the denominator increased, the fraction itself also increased. Wiest et al. (2015) agree with Stafylidou and Vosniadou (2004) that some students transfer whole-number ideas to fractions (i.e. larger numbers mean greater magnitude) and they focus on individual fraction components (numerator, denominator) rather than a fraction as a single entity.

The comparing and ordering of fractions on the basis of the numerators only or the denominators only, clearly showed that these learners lacked conceptual understanding of the role of the fraction numerators and fraction denominators, as well as the relationship to each other in a holistic manner (Gabriel et al., 2012; Weist et al., 2014). The responses of these learners suggested that they interpreted the given fractions as a pair of two whole numbers, instead of as parts of a whole. This study concurs with Amato (2005) who stipulates that many learners lacking conceptual understanding of the role of fraction numerators and denominators, tend to think of fractions as a pair of whole numbers, rather than as a single entity.

6.3.2.2 Adding fractions with different denominators inappropriately

The analysis of pre-intervention results showed that the majority of the learners in this study were able to appropriately add fractions with the same denominator (see table 4.4 in Chapter Four). The analysis of data also showed that half of the group of learners who participated in this study, used the lowest common denominator method inappropriately to add fractions with different denominators. They did this in three ways. The first inappropriate procedure involved adding numerators together and denominators together (see part 4.3.3.1.1 of Chapter Four). This procedure was used by L5 to conclude that 2/5+1/6=3/12. The second inappropriate procedure used involved adding fractions with a new common denominator and unchanged numerators (see part 4.3.3.1.2 of Chapter Four). This second procedure was used by three learners to conclude that 3/4+4/5=7/20. The third inappropriate procedure involved using inappropriate arithmetic manipulation to add fractions with same denominator and fractions with different denominators (see part 4.3.3.1.3 of Chapter Four). For instance, L2 and L10 used this method to conclude that 1/8+3/8=88/8 and 3/4+4/5=22/10 or 3/4+4/5=7/12.

This use of inappropriate procedures to add fractions agrees with Gabriel et al. (2012), who state that many learners seem to apply procedures such as the lowest common denominator method without fully understanding the underlying concepts. For instance, only one learner used fraction bars in the pre-test to visually represent the lowest common denominator method for adding fractions (see part 4.3.3.2.2 of Chapter Four). The inability of learners to visually represent the lowest common denominator method, suggested that many of these learners added fractions without understanding why they should maintain the denominator when adding fractions with the same denominator and why they should find the lowest common denominator when adding fractions with different denominators. Siegler et al. (2013) explain that the choice of inappropriate fraction arithmetic strategies when using the lowest common denominator method for adding fractions, is strongly influenced by the learners' constrained conceptual knowledge of fractions. In order to minimise this constraint, Siegler et al. (2013) and Siegler and Lortie-Forgues (2014) suggest that teaching of fractions should focus on improving learners' conceptual understanding of fraction magnitudes.

The next section discusses the changes in learners' conceptual understanding of and procedural fluency with fractions over the period of the teaching intervention.

6.4 CHANGES DURING THE TEACHING INTERVENTION

This section presents and discusses the research findings on the changes of the learners' conceptual understanding of and procedural fluency with fractions over the period of the teaching intervention (question 2). Generally, the findings suggested that the intervention did seem to help the learners in a number of different ways. These changes are presented in two subsections, namely: conceptual changes and procedural changes of fractions. The first subsection presents two themes of conceptual change, while the second subsection presents two themes of procedural change. Kilpatrick et al. (2001) state that the strands of mathematical proficiency are interwoven and interdependent; therefore, the procedural changes presented here are interwoven with the conceptual changes.

6.4.1 Conceptual changes of fractions

This subsection presents two themes of conceptual change, identified based on the analysis of post-intervention results of this study. The analysis of learners' responses in the post-test and post-interview, clearly indicated that at this stage, at least 10 of the learners who participated in this research showed mastery, by appropriately relating unit subdivisions to the denominator of fractions (see table 4.6 in Chapter Four). The appropriate relating of unit subdivisions to the denominator of fractions (see subsection 4.4.1.1 in Chapter Four) involved the following two themes. The first is that using area models and number lines seemed to help these learners to read fractions using appropriate names. The second theme is that using area models and number lines seemed to help these learners to identify the whole unit in fraction models and to develop a sense of the size of fractions in relation to one whole unit. These themes are discussed in detail below.

6.4.1.1 Reading fractions using appropriate names

The first conceptual change shown by these learners was reading fractions using appropriate fraction names. For instance, Table 4.6 shows that all the learners described fractions in questions 1.1 and 1.2 of the post-test, using appropriate names *"four ninths"* and *"eight elevenths"* respectively. The reading of fractions appropriately included the use of correct names for the number of equal-sized parts in the unit, which was not the case before the intervention where many used *"quarters"* to name fractions that were not quarters. The reading of fractions using appropriate names was developed during the intervention by asking learners to identify fractions of the area or length shaded, on area models and number lines. Way (2011) stresses the importance of using appropriate language when labelling fractions, as this practice allows learners to recognise fractions as numbers. Clarke et al. (2011) add that the use of appropriate fraction names could help learners to identify the digits that represent the number of parts and the size of parts. In the end, the use of appropriate fraction names enhanced learners' conceptual understanding of fractions by allowing learners to determine the size of fractions in relation to one whole unit (Way, 2011).

6.4.1.2 Identifying the whole unit in fraction models and developing a sense of the size of fractions in relation to one whole unit

The analysis of the post-intervention results showed that 10 out of the 12 learners demonstrated mastery in identifying the unit in area models and number lines having more than one unit. They did this by relating unit subdivisions to the denominator of fractions appropriately (see subsection 4.4.1.1 in Chapter Four). The post-intervention results also showed that the majority of learners had developed a sense of the size of fractions in relation to one whole unit, in three instances. For the first instance and for the first time, these learners located proper fractions between 0 and 1 on the number line showing more than one unit. For the second instance, these learners located improper fractions and mixed fractions after one on the number line. For the third instance, they identified fractional quantities less than one unit and fractional quantities greater than one unit from both fraction models, using appropriate proper fraction notation and improper fraction notation (or mixed fraction notation) respectively.

The extensive use of number lines to represent fractions in this study, may have had a number of benefits for the learners. As discussed by Clarke et al. (2011), the use of number lines helped the learners to see how whole numbers and fractions are related. The use of number lines allowed these learners to understand the relative size of fractions in relation to one whole and think of fractions as single numbers (Clarke et al., 2011; Pantziara & Philippou, 2012). In fact, the combined use of number lines and area models having more than one unit, allowed these learners to discover that improper fractions and mixed fractions are both representations for expressing fractional quantities bigger than one whole unit, while proper fractions are representations for fractional quantities less than one whole unit. This was indicated in the recall interview, when L6 explained that she learned that *"when you are going to draw a number line ... all proper fractions are before one and then between zero and one"* (Appendix H: lines 110-115). And when L11 explained that she *"learned that improper fractions are always written after one [on the number line]"* (Appendix H: line 128).

As discussed by Van de Walle et al. (2013), the use of two different models in this study to represent the same improper fractions was helpful for the learners to notice a pattern that actually explained the algorithm for converting between mixed fractions and improper fractions. The results of this study concur with the assertion of Cramer et al. (2008) that representations allow students to understand mathematical concepts. For instance, number lines and area models seemed to help the learners to understand the role of the fraction numerator and denominator. The results of a research study by Rau et al. (2013), showed that affording learners the experience to relate the numerator and denominator to each graphical representation, helps learners to benefit from the use of multiple graphical representations. Gray (2014) also believes that fraction models in the middle grades are a key to learners' success for conceptually understanding the symbols used to represent fractions, as well as for deepening learners' conceptual understanding of fractions.

6.4.2 Procedural changes

This subsection presents two themes relating to changes of procedural fluency with fractions, shown by the participants of this study. This subsection focuses on the types of procedures used for comparing, ordering and adding fractions, as well as on how these procedures were applied to solve fraction tasks of comparing, ordering and adding fractions. The two themes presented here could be related to the analysis of data presented in subsections 4.4.2.1 and 4.4.3.1 of Chapter Four. These procedural changes are interwoven with the conceptual changes in subsection 6.4.1. The first theme is that using area models and number lines seemed to help these learners to conceptually use equal fraction bars, equal number lines, benchmarking and rules for comparing and ordering fractions. The second theme is that using equal fraction bars to graphically illustrate addition of fractions seemed to help these learners to visually represent the lowest common denominator method and to recognise that only equally sized units can be counted together. These two themes are discussed in detail below.

6.4.2.1 Conceptually using equal fraction bars, equal number lines, benchmarking and rules to compare and order fractions

The analysis of the post-intervention results revealed that after the teaching intervention, the learners no longer compared fractions using inappropriate procedures such as the sizes of numerators and denominators separately, or inappropriate arithmetic manipulation, as they had in the pre-test. Instead, they used four, completely different, appropriate procedures to compare and order fractions, in the post-test and post-interview.

The first appropriate procedure applied to compare and order fractions after the teaching intervention was using equal number lines, while the second appropriate procedure was using equal fraction bars (see parts 4.4.2.1.1 and 4.4.2.1.2 of Chapter Four). In the post-test, eight learners represented the given fractions as the length shaded on distinct, equal fraction bars and equal number lines of one unit long and compared the fractions according to the length shaded, in such a way that the fraction which is represented by the bigger length shaded is the bigger one, while the fraction represented by the smaller length shaded is the smaller one. As discussed by Cramer et al. (2008) and Van de Walle et al. (2013), both fraction bars and number lines are good for helping learners to determine the relative size of fractions, based on the area or length shaded. In this study, both fraction bars and number lines of one unit long, helped learners to compare and order proper fractions appropriately (see parts 4.4.2.1.1

and 4.4.2.1.2 of Chapter Four). Cramer et al. (2008) and Van de Walle et al. (2013), indicate that the use of fraction models allows learners to clarify ideas that are often confused in a purely symbolic representation. For instance, using fraction models to graphically illustrate fractions, allowed the learners in this study, to determine the relative size of the fractions, instead of thinking of fractions as two unrelated whole numbers, and then use the size of either the numerators or denominators to compare and order fractions, like in the pre-test. Canterbury (2007) indicates that fraction models can be used in a number of ways to solve fraction tasks. In this study, fraction models helped learners to organise their mathematical thinking and reasoning. The fraction models also helped learners to obtain a visual representation of the task, to have a clear picture of the task and to check the accuracy of comparing and ordering fractions.

The third procedure applied to compare and order fractions in the post-test and post-interview is benchmarking (see part 4.4.2.1.4 in Chapter Four). Benchmarking refers to comparing and ordering fractions according to their closeness to fixed reference points on the number line, which can commonly be 0, 1/4, 1/2, 2/3, 3/4 and 1 (Clarke et al., 2011). For instance, in the post-interview, L3 used benchmarking to compare 5/8 and 3/10 first, by representing these fractions on distinct, equal number lines. L3 further explained that *"five eighths is greater than three tenths, because five eighths is closer to one, while three tenths is closer to zero"* (Appendix G: lines 117-118). In the post-interview, L6 also used equal number lines and benchmarking for comparing and ordering 3/5, 4/9 and 2/7. L6 explained that *"three fifths is closer to a whole which is one, four ninths is closer to a half and two sevenths is closer to zero… Since zero is the smallest and then comes a half and the whole, so they will just follow each other in the sequence. The smallest will be two sevenths and then four ninths and then three fifths"* (Appendix G: lines 162-166). This research study concurs with Clarke et al. (2011) who describes benchmarking as one of the creative strategies that most successful learners often use to compare the relative size of fractions. Eight out of the 12 learners in this study used benchmarking to compare fractions appropriately and most of these learners also made use of fraction number lines. The linear model is described to be good for visualising that between any two fractions there is always another fraction (Van de Walle et al., 2013). This form of thinking appeared to be a key factor for the learners who used benchmarking to compare fractions.

The fourth procedure applied after the teaching intervention, involved using two rules for comparing and ordering fractions with the same numerator and fractions with the same denominator (see part 4.4.2.1.3 in Chapter Four). The first rule is for comparing and ordering fractions with the same numerator which stipulates that *"when fractions have the same numerator, the fraction with the smallest denominator is the biggest one, while the fraction with the biggest denominator is the smallest one".* The second rule is for comparing and ordering fractions with the same denominator which stipulates that *"when fractions have the same denominator, the fraction with the biggest numerator is the largest one, while the fraction with the smallest numerator is the smallest one"* (see examples on the use of these rules in figure 4.18 of Chapter Four). These rules were used by eight learners in the post-test, in comparison to one learner in the pre-test. Wiest et al. (2012) stipulate that learners showing conceptual understanding of fractions develop more than one strategy of comparing fractions. In this study, the learners that used the above stated rules for comparing fractions, also made use of either equal fraction bars or number lines, to graphically illustrate the fractions (see figure 4.18 in Chapter Four). The combined use of the rules and fraction models showed meaningful understanding of the procedures for comparing and ordering fractions. The two rules were derived in class during the teaching intervention, to represent the pattern for comparing fractions with the same numerator and fractions with the same denominator.

6.4.2.2 Visually representing the lowest common denominator method and recognising that only equally sized units can be counted together

The analysis of learners' responses in the post-test and post-interview showed that the majority of learners continued to add fractions with same denominator appropriately. The analysis of data also revealed that few of the learners continued to use the lowest common denominator method inappropriately to add fractions with different denominators (see tables 4.5 and 4.8 in Chapter Four). The post-intervention results in Table 4.8 indicate an increase in the number of learners who used fraction bars to visually represent the lowest common denominator method, for adding both fractions with the same denominator and fractions with different denominators (see Figure 4.20 in Chapter Four for details). The use of both fraction bars and the lowest common denominator method to add fractions in the post-test and post-interview, strongly suggested that learners showed developing conceptual understanding and procedural fluency of the lowest common denominator method. Rau et al. (2013) underscore that fraction models play key complementary roles in enhancing conceptual aspects of fractions. Gray (2014) adds that the use of fraction models in the middle grades is a key to learners' success to conceptually anchor the algorithms used to work with fractions. In the context of this study, the use of fraction bars to visually represent the lowest common denominator method for adding fractions, appeared to help learners to understand that only equally sized units can be counted together. For instance, the analysis showed that every learner that used fraction bars to visually represent the lowest common denominator method, repartitioned fraction bars to create an equal number of same sized parts which was equivalent to the lowest common denominator for adding fractions.

6.5 POSSIBLE INFLUENCE OF THE TEACHING INTERVENTION

This section presents and discusses factors that possibly influenced the changes in learners' conceptual understanding of and procedural fluency with fractions as discussed in section 6.4. Four themes were identified as possibly influencing the conceptual changes and procedural changes. The discussion of these themes suggest the impact of the intervention on conceptual understanding and on procedural fluency through conceptual understanding, since the teaching intervention used a conceptual understanding approach, by relating the concepts and procedures to fraction models and fraction symbols when teaching fractions (see Appendix I).

The first theme, is that using area models and number lines to identify both fraction symbols and appropriate fraction names of the area shaded, seemed to help learners to see fractions as relational numbers. This theme is related to reading fractions using appropriate names in subsection 5.2.1. The second theme, is that prompting to partition whole units in area models and number lines, as a way of graphically illustrating fraction symbols, seemed to help learners to identify the whole unit in fraction models and to develop a sense of the size of fractions in relation to one whole unit. This theme relates to the results presented in subsections 5.2.2 and 5.2.3 of Chapter Five. The third theme, is that using area models and number lines to graphically illustrate fraction symbols, seemed to help learners to use equal fraction bars, equal number lines, benchmarking and rules for comparing and ordering fractions. This theme relates to the results presented in subsection 5.2.4 of Chapter Five. The fourth theme, is that using equal fraction bars to graphically illustrate fraction denominations, seemed to help learners to visually represent the lowest common denominator method for adding fractions and to recognise that only equally sized units can be counted together. This theme relates to the results presented in subsection 5.2.5 of Chapter 5. These themes are discussed in detail in subsections 6.5.1-6.5.4 below.

6.5.1 Identifying both fraction symbols and appropriate fraction names using fraction models, to see fractions as relational numbers

The first objective of the teaching intervention was to help learners to use correct language when reading fraction symbols. The teaching intervention at the onset, emphasised the reading of fraction symbols using appropriate names. This was done in four ways. Firstly, the teacher explained to the class that fractions are accurate measures of unknown quantities when the whole is divided into equal parts. Secondly, based on the use of fractions explained, learners were requested to identify fraction symbols for the area shaded on area models and number lines. Thirdly, learners were requested to read fraction symbols as one number, by using appropriate names and by avoiding the use of the word *"over"* when reading fractions. For instance, during the first lesson of the intervention, the teacher encouraged learners to "*avoid the use of the word 'over' when reading fractions because a fraction is one number and not two numbers"*. Fourthly, the teacher reinforced the reading of fractions as one number using appropriate fraction names. If the learner read the fraction symbols for the shaded area of a number line or an area model, as two whole numbers, the teachers used one of these four ways to reinforce the reading of fractions as one number:

- The teacher rejected the reading of fraction symbols as two whole numbers over one another by reminding the learner that a fraction is one number and not two numbers.
- The teacher would ask a learner to repeat what they said and the learner was then likely to see that he/she had read the fraction incorrectly.
- The teacher would say the correct name if a learner read a fraction as two whole numbers.
- The teacher would ask a learner or the class to confirm if the fraction was read correctly.

The techniques for reinforcing the reading of fractions as one number using appropriate names, were applied when asking learners to identify fraction symbols of the area shaded of area models or on number lines. As discussed by Way (2011), giving emphasis to the use of appropriate language when labelling fractions, allows learners to recognise fractions as numbers. On the contrary, the reading of fraction symbols as two numbers over one another, is regarded as very confusing, as it encourages learners to think of fractions as two unrelated whole numbers. The reading of fractions as single numbers appeared to be enhanced by relating fraction names to fraction models, including number lines, in this study. Pantziara and Philippou (2012) indicate that extensive use of number lines for fractions helps learners to understand the relative size of fractions and think of fractions as relational numbers. Research by Rau et al. (2013) suggest that the pairing of symbols such as fraction symbols with a single graphical representation like a number line, leads to better learning. Thus, the teaching of reading fractions as numbers in this intervention study was paired with fraction models, to enhance learning with understanding.

6.5.2 Prompting to partition models and graphically illustrating fraction symbols, to identify units in fraction models and develop a sense of the size of fractions in relation to one

Identifying the unit in models, and allowing learners to develop a sense of proper fractions and improper fractions in relation to one whole unit, were among the focus of the teaching intervention. To attain these objectives, the intervention used two methods of teaching, which are prompting and allowing learners to partition models using fraction denominators. This section presents the key activities that allowed learners to identify the units in fraction models and thereafter, the activities that enabled learners to develop a sense of the size of fractions in relation to one whole unit.

Learning to identify the unit in fraction models began, by asking the class to identify proper fractions using fraction bars, followed by using number lines. After the class identified the proper fractions, it was helpful to ask the class *"What does the denominator represent on a fraction bar or on a number line?"* Learners were encouraged to see that a denominator represents the number of parts that make up one whole fraction bar or one on a number line. On a number line, learners need to see that proper fractions are fractions less than one, by asking them a question such as *"This fraction, is less than what?"* In response, they would either mention the denominator or one. It was then helpful to ask learners to write one on the number line as a fraction, with unit subdivisions. Van de Walle et al. (2013) suggest that teachers need to ask questions such as those ones given above, to help learners to make sense of fraction symbols.

Another important activity which appeared to help learners to identify the unit on number lines, was asking learners to locate a proper fraction on number lines, simply by partitioning the unit using the denominator to work out unit subdivisions. This study agrees with Cramer (2008) that the majority of learners would require extended periods of time and lots of practice, if they are to fully grasp the idea of fractions as numbers. Partitioning of units on number lines using denominators was not an easy task for the learners, especially to write one as a fraction with unit subdivisions. In this regard, it was helpful for the teacher to count unit subdivisions with learners, for the learners to see one as a fraction with unit subdivisions. After learners were able to partition units on number lines and to relate unit subdivisions to one, they were asked to locate proper fractions and improper fractions (or mixed fractions) respectively, on number lines. The teacher also extensively focused his teaching on learners to partition fraction models by themselves to enable them to think creatively and visualise unit subdivisions in models. This intervention study agrees with Way (2011), that learners need to partition fraction models instead of using pre-partitioned models, to enable them to think creatively and visualise unit subdivisions in fraction models.

To develop learners' sense of the size of fractions in relation to one whole unit, it was helpful to ask learners first to identify improper fractions on pre-partitioned number lines. Then, learners were asked to represent the same improper fractions using circles. This helped learners to discover that improper fraction notation and mixed fraction notation are representations of fractional quantities greater than one whole unit. Representing the same improper fraction using number lines and circles, allowed learners to relate whole numbers on number lines to number of fully shaded circles. As suggested by Clarke et al. (2011), the use of both number lines and area models appeared to have contributed effectively to developing the conceptual meaning of improper fractions and their equivalences. Once learners were comfortable to represent improper fractions (or mixed fractions) and proper fractions, it was helpful to ask learners to list all properties they had learned of proper fractions and improper fractions. Learners were likely to list only one property of proper fractions, as fractions whose numerators are smaller than the denominators, while improper fractions, as fractions whose numerators are bigger than the denominators. It is actually the responsibility of the teacher, to help learners to describe proper fractions in relation to one unit, as fractions less than one unit and fractions whose numerators are always smaller than the denominators. In the same way, learners could also describe improper fractions as another form of mixed fractions, as fractions greater than one unit and whose numerators are greater than the denominators. It is also important for the teacher to help learners see the relationships between the properties of each type of fractions by referring to graphical illustrations of fraction symbols on fraction models.

As discussed by Pantziara and Philippou (2012), developing learners' ability to identify fractions as points on number lines, allows learners to think of fractions as single numbers. For instance, in this study, learners had developed a sense of the size of fractions in relation to one whole unit, by regarding proper fractions as fractions less than one whole unit and improper fractions or mixed fractions as fractions greater than one whole unit. The use of two fraction models also helped learners to make connections among models. This study agrees with Gray (2014) and Rau et al. (2013) who state that helping learners to make connections among models, may deepen their conceptual understanding of fractions, especially identifying the unit in models (Gray, 2014; Rau et al., 2013). This study also concurs with Rau et al. (2013) that extensive use of prompts to support understanding of graphical representations is an effective way of broadening learners' understanding of fractions and helping learners to conceptually relate the concepts of the numerators and denominators to each representation. In the context of this study, the use of prompts helped learners to see how whole numbers are fractions with unit subdivisions, to see how mixed fractions and improper fractions are related, as well as to relate unit subdivisions to the denominator of fractions and vice-versa. In addition to the use of prompts and as suggested by Clarke et al. (2011) and Siegler et al. (2010), this intervention study made use of one-to-one interviews with the learners, to gain insight into their thinking, to help learners to make sense of fractions, and to confirm their understanding.

6.5.3 Graphically illustrating fraction symbols using models, to use equal fraction bars and number lines, benchmarking and rules for comparing

The intervention study used a non-traditional teaching approach to help learners to compare and order fractions, namely: using equal fraction bars and equal number lines. The fraction bars and number lines were used to graphically illustrate a pair and triplet of fraction symbols, so that learners could visually determine the relative size of the fractions they were comparing. Firstly, this section presents how the fraction bars were used to help learners to compare and order fractions followed by the use of number lines to compare and order fractions.

For the use of fraction bars, learners used the shaded area of equal pre-partitioned fraction bars to compare sizes of fractions. To help learners to deduce the patterns of comparing fractions using fraction bars, the teacher used three different worksheets and every worksheet had examples of one type of fractions (i.e. one worksheet had fractions with same denominator, another worksheet showed fractions with same numerator, and the other worksheet had fractions with different numerators and denominators). The teacher assisted learners to compare fractions appropriately by demonstrating and prompting learners to visually illustrate fraction symbols, using fraction bars. Learners used fraction bars to compare and order fractions appropriately, using the shading of areas to determine the relative size of fractions. Learners, with the help of the teacher were able to deduce two rules for comparing fractions with the same denominator and fractions with the same numerator by answering the question: "*What did you find special about comparing the fractions [in each worksheet]?*" The first rule states that "*for fractions with the same numerator, the bigger fraction is the one with the smaller denominator*" and the second rule states that "*for fractions with the same denominator, the bigger fraction is the one with the bigger numerator*".

Number lines were used by learners to compare fractions, first by locating fraction symbols on number lines and then ordering fractions using benchmarks of 0, 1/4, 1/2, 3/4 and 1. Learners were asked to write explanations of how they used number lines to compare fractions. At first, learners found the writing of explanations very difficult, but most mastered the use of benchmarking by the end of the eighth lesson. However, by the end of this intervention, three learners (L4, L5 and L8) had not changed, as they were still comparing and ordering fractions using the sizes of the denominators, despite the efforts of introducing them to the use of fraction bars and number lines (see Figure 5.12). The analysis of this research study suggests that both number lines and area models helped the learners to compare the relative size of fractions. This intervention study agrees with Van de Walle et al. (2013) who state that both number lines and fraction bars help learners to compare the relative size of fractions. This intervention study also concurs with Way (2011) and Mack (1990) that fraction concepts develop over time and teachers need to be patient with the learners to accord them enough time to learn the fraction concepts.

6.5.4 Graphically illustrating fraction denominations using equal fraction bars to recognise that only equally sized units can be counted together

The intervention study used equal fraction bars to graphically illustrate fraction denominations and the lowest common denominator method for adding fractions. The objective of using fraction bars to illustrate the lowest common denominator method for adding fractions, was for the learners to recognise that the rule of finding the common denominator whenever adding fractions, is directly linked to the rule of counting, which is simply that one can only count together equally sized objects.

The teaching of adding fractions during the intervention, followed a particular sequence of actions. Firstly, the teacher and the class worked together on the chalkboard, to demonstrate adding of fractions using the rules of the lowest common denominator method. Then, the teacher and the class used fraction bars to illustrate fraction denominations and to repartition fraction bars to create equal parts in every whole. Repartitioning of fraction bars was necessary to make sure that the parts in every whole were equally sized, before they could be counted together to find the sum of fractions. The teacher also explained to the class that if the parts of fraction bars showing addend fractions are not of equal size, they need to be repartitioned to create the same total number of equally sized parts in every fraction bar and that this was the reason why learners have to find the common denominator, if the denominators of addend fractions were different. All learners were amazed to learn the use of fraction bars to show when there is a need to find the lowest common denominator and when there is not, as well as that the lowest common denominator is linked to the idea of counting equally sized units. As discussed by Gabriel et al. (2012), the learners appeared to have been applying the lowest common denominator method when adding fractions without understanding the underlying concepts. However, the increase in the number of learners who used repartitioning of fraction bars to graphically illustrate the lowest common denominator method and appropriately added fractions with different denominators, suggested that these learners understood that the lowest common denominator is linked to the idea of counting equally sized units only.

Some learners found it easy to illustrate fraction denominations and the process of adding fractions, using the lowest common denominators, while some learners did not know how to proceed after representing the addend fractions. The teacher noted that most of the learners who did not know how to illustrate addition of fractions using fraction bars, were those who did not follow the instruction of working out the sum of fractions, before attempting to illustrate the process of adding using fraction bars. This intervention study concurs with Way (2011), Clarke (2011) and Mack (1990) that fraction concepts develop over time and therefore teachers need to be patient and passionate about teaching fractions using models, for the learners to learn fraction concepts. In this study, teaching of fraction models to represent addition of fractions was conducted within one lesson only, and the limited time of exposing learners to the use of models in this domain seemed to contribute to low conceptual change of adding fractions. Cramer et al. (2008) indicate that the majority of learners require extended periods of time and lots of practice if they are to fully grasp the idea of using area models to show the process of adding fractions.

6.6 CONCLUSION

This chapter discussed the research findings of this study in relation to the literature review in Chapter Two. The discussion of findings had three areas of focus, namely: the nature of conceptual and procedural difficulties displayed by the Grade 8 learners before the teaching intervention; the conceptual changes and procedural changes of fractions displayed by the learners after the teaching intervention; and the factors that influenced the changes of learners' conceptual understanding of and procedural fluency with fractions over the period of the teaching intervention.

CHAPTER 7

CONCLUSION

7.1 INTRODUCTION

This study investigated the use of models to develop Grade 8 learners' conceptual understanding of and procedural fluency with fractions. An intervention programme for a sample of 12 Grade 8 learners was developed, implemented, and evaluated. The use of an intervention in this study qualifies this research as an intervention study and it had three areas of focus. The first was to investigate and establish the nature of these learners' conceptual understanding of and procedural fluency with fractions, before the teaching intervention, by analysing learners' responses in the pre-test and pre-interviews. The second was to investigate and establish the conceptual changes and procedural changes in fraction work of these learners after the teaching intervention, by analysing learners' responses in the post-test, post-interviews and recall interviews. The third was to investigate and establish the possible influence of the teaching intervention on these changes.

The analysis of the data of this research provided insight into the difficulties the selected Grade 8 learners faced with learning fractions and the opportunities that the use of models may provide for developing learners' conceptual understanding of and procedural fluency with fractions, particularly with respect to fraction concepts, comparing and ordering fractions, and addition of fractions.

This chapter concludes the study by presenting themes and insights that emerged through the data analysis of this intervention study. The structure of this chapter is comprised of:

- A summary of the findings;
- Significance of this intervention study;
- Limitations and challenges of the study;
- Reflections and recommendations, and lastly;
- Areas of further research.

7.2 SUMMARY OF FINDINGS

The summary of findings for this study is organised according to the three areas of focus, and presented in three subsections, namely: before the intervention, after the intervention and the teaching intervention, respectively.

7.2.1. Before the intervention

This subsection presents the summary of findings for the first research question about the nature of the selected Grade 8 learners' conceptual understanding of and procedural fluency with fractions before the teaching intervention. The investigation of the learners' knowledge of fractions prior to the teaching intervention, suggested that these learners had difficulties in their conceptual and procedural engagement with fraction models and fraction symbols. The difficulties displayed by the learners were classified in two groups, namely: one relating to conceptual understanding and the other to procedural fluency with fractions. Although the findings were classified in two groups, the themes of procedural difficulties are related to conceptual misunderstandings.

7.2.1.1 Themes of conceptual understanding of fractions

a) *Reading fractions using inappropriate names*

The analysis of the pre-intervention results showed that the majority of learners read fractions using inappropriate names. This occurred in two ways. The first was to generalise the use of the word *"quarters"* to name fractions that were not quarters. For instance, they read 5/12 either as *"five quarters"* or *"five quarters of twelve"*. The second inappropriate naming convention was to read fractions as two unrelated whole numbers *"over"* one another. For instance, some learners read 5/12 as *"five over twelve"*. Clarke et al. (2011) describe the reading of fractions as two whole numbers over one another as confusing, saying that it makes it difficult for the learners to tell which of the digits of the fraction symbol represents the number of parts or the size of parts in the unit. In this study, it appeared to be difficult for the learners to identify the whole unit in the models (see the next theme for details). Vatilifa (2012) indicates that the use of incorrect terminology such as '1 over 4' or '1 out of 4' when teaching fractions in Namibia is common. According to Vatilifa (2012), fractions in Namibia are taught only through symbolic representations of abstract fraction concepts.

b) Not identifying the whole unit in the models and therefore identifying fractions using inappropriate fraction symbols

The analysis of pre-intervention results showed that 11 out of the 12 learners identified fractions represented by a single area model, using appropriate fraction symbols. These results also revealed that most learners had difficulties identifying the unit on number lines showing more than one unit and number lines of one unit long, as well as circles showing more than one unit. For instance, in the pre-test, all learners located a proper fraction 3/4 between 3 and 4. Mitchell and Horne (2011) and Larson (1987) indicate that many learners commit this error of locating fractions less than one unit after one, by disregarding the scaling and treating the whole number line as a unit.

In the second instance of not identifying the unit, the majority of learners identified fractions represented by area models and number lines showing more than one unit, using inappropriate fraction symbols (see paragraphs 7-11 of subsection 4.3.1.1). These learners did not use improper fraction notation or mixed fraction notation to identify these fractions.

In the third instance, some of the learners identified fractions on number lines one unit long using decimal notation, whole numbers and inappropriate fraction notation such as *1/(number of shaded parts)* and *(number of shaded parts)/1*. Mitchell and Horne (2011) and Van de Walle et al. (2013) state that the use of incorrect fraction notation indicates that learners find difficulty in working with area models showing more than one unit and fraction number lines. In a nutshell, the conceptual engagement of fraction models and use of fraction symbols demonstrated by the learners of this research study is described by Clarke et al. (2011) as showing a limited part-whole interpretation of fractions. The learners' responses suggested that their experience of fraction models is limited to a part-whole interpretation of fractions of a single area model, whereby learners only think of fractions as shaded parts over the total number of pieces in a single, whole unit. The analysis of Namibian mathematics textbooks for Grade 5 and 6 presented in section 2.2 of Chapter Two, showed that the part-whole interpretation of fractions is generally related to only single wholes such as circles, rectangles and fraction charts. Amato (2005) believes that learners whose experience of fractions is limited to part-whole interpretation of fractions of a single whole, generally find it difficult to identify units in the area models and number lines showing more than one unit.

7.2.1.2 Themes of procedural fluency of fractions

a) *Using the sizes of the numerators and denominators separately*

The analysis of the pre-intervention results showed that the majority of learners had difficulties interpreting fractions as relational numbers; instead, they used at least one of the following two inappropriate procedures to compare and order fractions. The first involved comparing and ordering fractions using the sizes of the denominators only, in two ways (see part 4.3.2.1.1 of Chapter Four). One method used the size of the denominators to compare fractions, so that the bigger fraction was the one with the bigger denominator and the smaller fraction was the one with the smaller denominator. Four learners in the pre-test used this method to conclude that 4/5<4/6; 1/2<3/8; 4/12>1/3; 1/2<1/3<1/4; and 2/3<5/5<1/12. The other method used the sizes of denominators to compare fractions so that the smaller fraction was the one with a bigger denominator and the bigger fraction was the one with a smaller denominator. Two of the learners used this second method in the pre-test to conclude that 4/12<1/3 and 1/12<5/5<2/3.

The second procedure used to compare and order fractions inappropriately in the pre-test and pre-interviews involved using the numerators as divisors (see part 4.3.2.1.2 of Chapter Four). This procedure was used by four learners to conclude that 2/7>3/7; 4/5<4/6; 4/12<1/3; 8/10<5/10<3/10; and 5/5<2/3<1/12 respectively. The comparing and ordering of fractions on the basis of the size of the denominator only or the numerator only is similar to the research findings of Stafylidou and Vosniadou (2004). Wiest et al. (2015) and Stafylidou and Vosniadou (2004) suggest that students commit this error by transferring whole number knowledge to fractions (e.g. larger numbers mean greater magnitude) and by focusing on individual fraction components (numerator, denominator) rather than on a fraction as a single entity.

b) *Adding fractions with different denominators inappropriately*

The analysis of pre-intervention results showed that the majority of the learners in this research study added fractions with the same denominator appropriately. The analysis of data also showed that half of the group of the learners who participated in this study used the lowest common denominator method inappropriately to add fractions with different denominators, in three ways. The first inappropriate method was by adding numerators together and denominators together, e.g. 2/5+1/6=3/12. The second inappropriate method was by adding fractions with a new common denominator and unchanged numerators, e.g. 3/4+4/5=7/20. The third inappropriate method was using inappropriate arithmetic manipulation to add fractions, e.g. 1/8+3/8=88/8; 3/4+4/5=22/10; 3/4+4/5=7/12. Gabriel et al. (2012) suggest that many learners use the lowest common denominator method without fully understanding the underlying concepts. Siegler et al. (2013) believe that the use of inappropriate fraction arithmetic strategies for adding fractions is strongly influenced by the learners' constrained conceptual understanding of fractions.

7.2.2. After the intervention

This subsection presents the summary of findings for the second research question about the changes of learners' conceptual understanding and procedural fluency of fractions after the teaching intervention. The research findings suggest that the intervention did seem to help these learners, as they showed developing conceptual understanding and procedural fluency of fractions in a number of ways. The conceptual and procedural changes of the learners are presented in two categories, namely: themes of conceptual changes and procedural changes of fractions, respectively. Although the findings are presented in two categories, the nature of procedural changes displayed by the learners is interwoven with the conceptual changes.

7.2.2.1 Themes of conceptual changes of fractions

a) *Reading fractions using appropriate names*

Reading fractions using appropriate names was one of the capacities attained by all learners after the teaching intervention in comparison to the majority of learners who read fractions using inappropriate fraction names before the teaching intervention. For instance, in the post-test, all learners read fractions using fraction names such as *"four ninths"* and *"eight elevenths"*. Way (2011) stressed the importance of using appropriate language when labelling fractions, as this practice allows learners to recognise fractions as numbers and to identify the digits that represent the number of parts and size of parts in the whole unit.

b) *Identifying the whole unit in the fraction models and developing a sense of the size of fractions in relation to one whole unit*

Ten out of twelve learners demonstrated mastery for identifying the unit in the area model and number lines showing more than one unit and number lines of one unit long. They demonstrated this mastery by relating unit subdivisions to the denominator of fractions appropriately (see subsection 4.4.1.1 in chapter 4). The results also showed that the majority of learners had developed a sense of the size of fractions in relation to one whole unit in three instances. In the first instance and for the first time, these learners located proper fractions between 0 and 1 on the number line showing more than one unit. In the second instance, these learners located improper fractions and mixed fractions after one on the number line. In the third instance, they identified fractional quantities less than one unit and fractional quantities greater than one unit from both fraction models using appropriate proper fraction notation and improper fraction notation (or mixed fraction notation) respectively. The combined use of the number lines and the area model showing more than one unit in this study allowed the learners to discover that improper fractions and mixed fractions are both representations for quantities bigger than one whole unit, while proper fractions are representations for fractional quantities less than one unit. In particular, the use of number lines helped the learners in this study to see how whole numbers and fractions are related; to understand the relative size of fractions in relation to one whole and to think of fractions as single numbers, as suggested by Pantziara and Philippou (2012) and Clarke et al (2011).

7.2.2.2 Themes of procedural changes of fractions

a) *Conceptually using equal fraction bars, equal number lines, benchmarking and rules to compare and order fractions*

This research study suggested that after the teaching intervention, the majority of learners no longer compared and ordered fractions inappropriately using the sizes of numerators only or denominators only. Instead they used four, completely different and appropriate procedures to compare and order fractions in the post-test and post-interviews. The first two appropriate procedures were using equal number lines and using equal fraction bars respectively. The majority of learners applied the first two procedures. As discussed by Cramer et al. (2008) and Van de Walle et al. (2013), both fraction bars and number lines helped the learners to determine the relative size of fractions, based on the area or length shaded. The third appropriate procedure applied in the post-test was benchmarking, whereby learners compared and ordered fractions according to their closeness to fixed reference points on the line, including 0, 1/4, 1/2, 2/3, 3/4 and 1. The fourth appropriate procedure applied to compare and order fractions was using two rules for comparing and ordering fractions with the same numerator and fractions with the same denominator. The rules were: *"When fractions have the same numerator, the fraction with the smallest denominator is the biggest one, while the fraction with the biggest denominator is the smallest one"* and *"When fractions have the same denominator, the fraction with the biggest numerator is the largest one, while the fraction with the smallest numerator is the smallest one"*.

b) *Visually representing the lowest common denominator method and recognising that only equally sized units can be counted together*

The research findings of the post-intervention results showed an increase in the number of learners who used fraction bars to visually represent the lowest common denominator method. The analysis also showed an increase in the number of learners who applied the lowest common denominator method appropriately to add fractions with different denominators. Gary (2014) and Rau et al. (2013), both stress the importance of using fraction bars in the middle grades as a key to learners' success to conceptually anchor the algorithms used to work with fractions. In this study, analysis of post-intervention results suggested that the use of fraction bars to visually represent the lowest common denominator method for adding fractions, appeared to help learners to recognise that only equally sized units may be compared by counting, which serves as the reason for calculating the lowest common denominator when adding fractions with different denominators. Learners demonstrated this understanding by repartitioning fraction bars to create equal parts in every whole fraction bars of addend fractions.

7.2.3. The teaching intervention

This subsection presents the summary of findings for the third research question about possible factors leading to the changes of learners' conceptual understanding and procedural fluency of fractions as discussed in subsection 7.2.2. The analysis of the teaching intervention suggested four factors that may have influenced conceptual changes and procedural changes, and these factors are briefly discussed below. These findings suggest the impact of the intervention on conceptual understanding and on procedural fluency through conceptual understanding, since the teaching intervention used a conceptual understanding approach by relating the concepts and procedures to the fraction models and fraction symbols, to teach both concepts and procedures related to fractions.

a) *Identifying both fraction symbols and appropriate fraction names using the fraction models, to see fractions as relational numbers*

The teaching intervention reinforced the reading of fraction symbols as one number by encouraging learners to use appropriate fraction names and discouraging learners from reading fraction symbols as two whole numbers. The teacher reinforced the use of appropriate fraction names by asking learners to read the fraction symbols of the area or length shaded on the area model and number lines. As discussed by Way (2011), giving emphasis to the use of appropriate language when labelling fractions allows learners to recognise fractions as relational numbers.

b) *Prompting to partition models and graphically illustrating fraction symbols, to identify units in fraction models and develop a sense of the size of fractions in relation to one*

The intervention used two methods of teaching which were prompting and allowing learners to partition the fraction models using fraction denominators. One of the important activities that appeared to help learners to identify the unit on the number line was asking learners to locate proper fractions on a number line (or represent proper fractions using area model) and when learners had to partition the unit length or whole area model using the denominator to work out the unit subdivisions. The partitioning of the unit length or unit area appeared to help learners to be able to relate unit subdivisions to the fraction denominator. In addition, the intervention developed learners' sense of the size of fractions in relation to one whole unit, by asking learners to represent and identify proper fractions and improper fractions on the number line and area model showing more than one whole unit. The teacher also asked the learners to describe proper fractions and improper fractions in relation to one whole unit. These efforts helped learners to recognise proper fractions as fractions less than one unit, and improper fractions and mixed fractions as fractions greater than one whole unit.

c) *Graphically illustrating fraction symbols using models to use equal fraction bars and number lines, benchmarking and rules for comparing*

The intervention study extensively used equal fraction bars and equal number lines to help learners to compare and order fractions. Fraction bars and number lines were used to graphically illustrate a pair or triplet of fraction symbols, so that learners could visually determine the relative size of fractions they were comparing. The intervention introduced the use of fraction bars to compare fractions by using three different worksheets and every worksheet had only fractions of the same type. The use of different worksheets for different types of fractions helped learners deduce two patterns (rules) of comparing fractions, namely: *"For fractions with the same numerators, the bigger fraction is the one with the smaller denominator"* and *"For fractions with the same denominators, the bigger fraction is the one with the bigger numerator".* In addition, number lines were used to help learners to compare and order fractions by locating fraction symbols on the number lines. Each time learners were using the number lines, the teacher asked learners to use benchmarks of 0, 1/4, 1/2, 2/3, 3/4 and 1 to write explanations of how they used the numbers on the number lines to compare and order fractions. The practice of writing explanations helped learners to learn and used benchmarking as another method of comparing fractions. Despite the extensive use of prompting and fraction models to compare and order fractions, three learners (L4, L5 and L8) did not change as they continued to compare and order fractions using the sizes of denominators, although they could illustrate fraction symbols on the fraction models. They seemed to require more time to learn the fraction concepts.

d) *Graphically illustrating fraction denominations using equal fraction bars to recognise that only equally sized units can be counted together*

The teaching intervention used fraction bars to graphically illustrate fraction denominations and the lowest common denominator method for adding fractions. Using fraction bars to illustrate fraction denominations of fractions with the same denominator and fractions with different denominators, helped the learners to see that when the parts of addend fraction bars are the same size, they can count easily to find the sum, while when the parts of addend fraction bars are of different size, they needed to repartition these fraction bars to create equal sized parts in order to find the sum. The illustration of the lowest common denominator method using fraction bars helped learners to recognise that only equally sized units can be counted together.

7.3 SIGNIFICANCE OF THIS INTERVENTION STUDY

The significance of this study is three-fold. Firstly, this study investigated the nature of Namibian Grade 8 learners' conceptual understanding and procedural fluency of fractions before the teaching intervention and the findings indicated that these learners displayed a number of difficulties in thinking of fractions as numbers and in comparing and adding fractions. These findings will help mathematics teachers to be aware of difficulties and misconceptions of fractions shown by the learners and to plan their fraction instructions to address these shortcomings.

Secondly, the findings of this study suggest that the combined use of number lines and area models showing one unit and showing more than one unit, helped learners to be able to identify the unit, to develop the sense of the size of fractions in relation to one, to compare and order fractions appropriately as well as to understand the conceptual meanings of finding the lowest common denominators of adding fractions. These findings suggest that these are opportunities for more effective fraction instruction in Namibian teaching and that such instruction may develop and improve their learners' conceptual and procedural knowledge of fractions.

Finally, the findings of this study may help mathematics textbook writers in Namibia to include number lines and area models showing more than one unit in the school textbooks.

7.4 LIMITATIONS AND CHALLENGES OF THE STUDY

Maree (2015) stresses that "the goal of qualitative research is not to generalise findings across a population, [but] … to provide understanding from the participants' perspective" (p. 115). Since this research project was exploratory research, its findings should not be generalised across all Grade 8 learners in Namibia because its sample was limited to 12 learners from one selected secondary school. Yet, its findings do provide understanding from the participants' perspective and could be useful in classroom situations where teachers seek to improve their teaching practices on fractions.

The findings of this research project are limited to the observations, perception and interpretations of myself, a novice researcher, who designed and implemented the research instruments. However, I capitalised on my teaching experience of mathematics, the teaching resources that were at my disposal, as well as the guidance received from my research supervisor to increase the trustworthiness of the research design and research instruments according to the research goals of this study.

Another limitation is that, even though it seems reasonable that the influence of the intervention had a lot to do with the learner changes, the design of the study can only suggest this. The study does not provide verifiable evidence that this was the result of the intervention. This would require a different research design.

7.5 REFLECTIONS

This research project was very educative and inspiring yet challenging at times. In this section, I will share my research experience by looking at how I became transformed and learned through research.

Firstly, I would describe the process of conducting research as a *'no recipe approach'*. I have learned that every research project is different from other research, in ways that include its research methodology, research goals, and data analysis techniques, through data presentation styles to its findings. The crafting of the research questions, research proposal and this book's chapters demanded applying creativity and reflexivity of the level best from both myself and my research supervisor.

I found the findings of this research very inspiring and yet it left me with many unanswered questions. I have learned in this project that the teaching and learning of fractions may be great fun, enjoyable and meaningful to the learners, by means of using fraction models. For instance, I have learned that the use of number lines and area models showing more than one unit can help learners to develop a sense of the size of fractions and help learners to justify the properties of proper fractions and improper fractions, as well as to compare and order fractions appropriately using a variety of methods. I have also learned that developing learners' conceptual understanding of fractions requires patience, passion and creativity from the teacher. Despite the conceptual changes and procedural changes that appeared to be attained, I am fearful that the status quo of teaching fractions will continue in Namibian classrooms. It is my wish that Namibian mathematics teachers and mathematics curriculum developers learn from the findings of this study, in order to add to their fraction knowledge and improve the teaching and learning of fractions. Finally, I learned that fractions are very complex and difficult to understand, as fractions are defined by many constructs and many representations. Therefore, I think it is important for mathematics teachers to acquaint themselves with these constructs and representations.

7.6 RECOMMENDATIONS

Based on the findings of this research, this study recommends the following ideas to enhance learners' conceptual understanding and procedural fluency of fractions.

- Mathematics teachers could encourage the use of correct language when reading fractions so that learners think of fractions as relational numbers rather than as two whole numbers over one another.
- Mathematics teachers and mathematics curriculum developers could include use of number lines and area models showing more than one unit to help learners to identify the units in the models and to develop the sense of the size of fractions in relation to one whole unit.
- Mathematics teachers could use prompting and partitioning of fraction models to help learners to conceptually use equal fraction bars, equal number lines, benchmarking and rules of comparing fractions with the same denominator and same numerator to compare and order fractions appropriately.
- Mathematics teachers could use fraction bars to graphically illustrate fraction denominations and visually represent the lowest common denominator method for adding fractions to help learners to recognise that only equally sized objects can be counted together.
- Finally, mathematics teachers could also be aware of the conceptual and procedural difficulties learners are faced with, if learners' conceptual understanding and procedural fluency of fractions is not well developed.

7.7 AREAS OF FURTHER RESEARCH

Based on the insights of the findings and limitations of this study, this study suggests the followings:

- That a similar study with a larger sample size and a different design be carried out across different regions of Namibia, in order to provide a more complete picture which could lead to the generalisation of a case study's results.
- That since this study only focused on one grade, I think similar studies with junior primary learners, senior primary learners, or secondary school learners may provide important findings to inform the designing of the fraction learning content in the Namibian curriculum.

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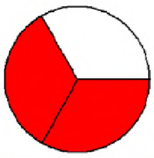
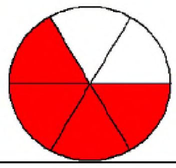
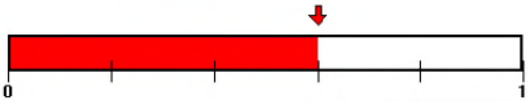
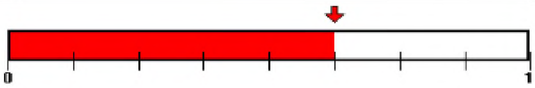
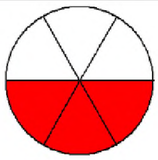
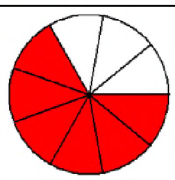
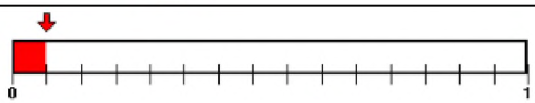
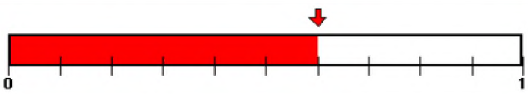
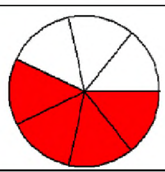
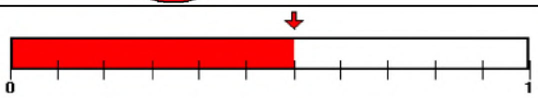
APPENDICES

APPENDIX A: WORKSHEETS OF TEACHING INTERVENTION

Activity 1: Identifying and naming of fractions Name:

This activity is designed to help learners to use the area model (circle) and number line to identify and write fractions correctly. In both models, the shaded part(s) should be used to write a fraction. A fraction helps us to determine the size of the whole that is shaded. The size of the shaded part of a whole CAN ONLY be determined once the whole is divided into equal parts. In your group of three, complete the table. The teacher will use number one as an example to show you how to complete the table correctly.

Partitioned circles and number lines	Number of shaded parts	Number of total parts in a whole	Fraction (symbol)	Fraction in words
1. 				
2. 				
3. 				
4. 				
5. 				
6. 				
7. 				
8. 				
9. 				
10. 				

Worksheet adapted from http://www.visualfractions.com

Use your answers in the table above to answer the following questions:

a) In which column can you find only the numerators of the fractions?

....

b) In which column can you find only the denominators of the fractions?

....

c) All fractions in this activity belong to one type of common fractions. Identify the type of these common fractions.

....

d) Write down the property (feature) that you used to get your answer in *part c.*

....

....

e) Knowing the denominator of a fraction is a very important aspect of understanding its value. What does the denominator tell us about a fraction?

....

....

....

Name:

Worksheet 2A: Representing the same fraction with multiple models

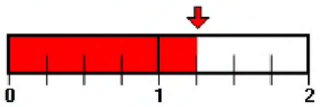
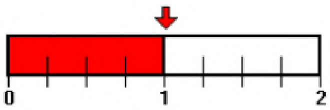
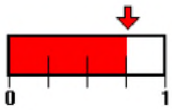
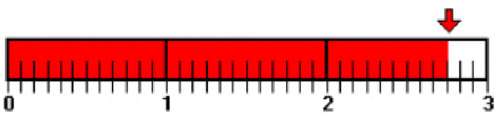
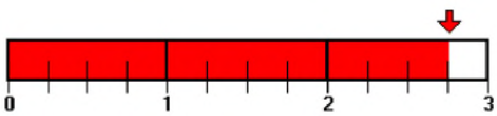
Representing fractions on the number lines as shaded parts of a circle's area

This activity is designed to help learners to realise and appreciate that a fraction can be represented using more than one model. To be specific, at the end of this activity, learners should be able to recognise that a fraction on a number line can be represented using an area model (circle) and vice-versa. In addition, this activity aims to help learners develop a more flexible and mathematically sound understanding of part-whole relations of fractions.

Complete the activity by:

a) Drawing the appropriate number of circles using a compass.
b) Dividing the circles based on the partitioning used on the number line.
c) Shading parts of circles to represent the fraction on a number line.
d) Writing the fraction represented in both models.

NB: Use one circle to represent one as a unit on a number line.

Number Line	Circle	Fraction
1. 		
2. 		
3. 		
4. 		
5. 		

Worksheet adapted from http://www.visualfractions.com

Describe how you used the partitioning and shading of the number line to represent a fraction as shaded area of a circle.

……
……
……
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Name:

Worksheet 2B: Representing fractions on circles as shaded length on a number line

Complete the activity by:

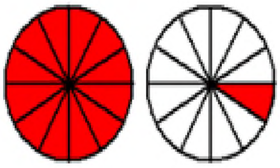
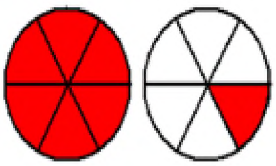
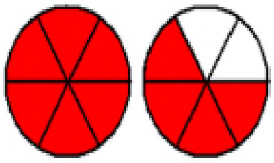
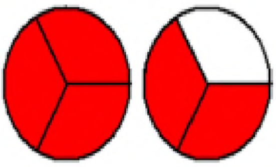
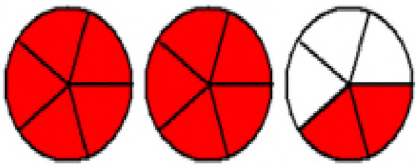
a) Drawing the appropriate length of a number line using a ruler.

b) Dividing the number line based on the partitioning used on a circle.

c) Shading the appropriate length on a number line to represent the fraction shown on circles.

d) Writing the fraction represented in both models.

NB: One circle is equal to one on a number line.

Circle	Number line	Fraction	Alternative fraction
1. 			
2. 			
3. 			
4. 			
5. 			

Work sheet http://www.visualfractions.com

Describe how you used the partitioning and shading of circle's area to show the same fraction on a number line.

....

....

....

....

....

e) Some fractions can be written in the alternative form, for instance $\frac{7}{5}=1\frac{2}{5}$; $4\frac{5}{6}=\frac{29}{6}$ or $3=\frac{18}{6}$.

Write down the alternative form of fractions in part d) above in the identified column.

f) Use your answers in the table above to help you to answer the following questions.

1. Describe the process of getting a mixed number.
   Hint: Start by defining a mixed number.

   ....
   ....
   ....
   ....
   ....

2. Describe the process of obtaining improper fractions.
   Hint: Start by defining improper fractions

....
....
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....
....

3. When is it possible for you to give your answer as improper fraction or mixed number?

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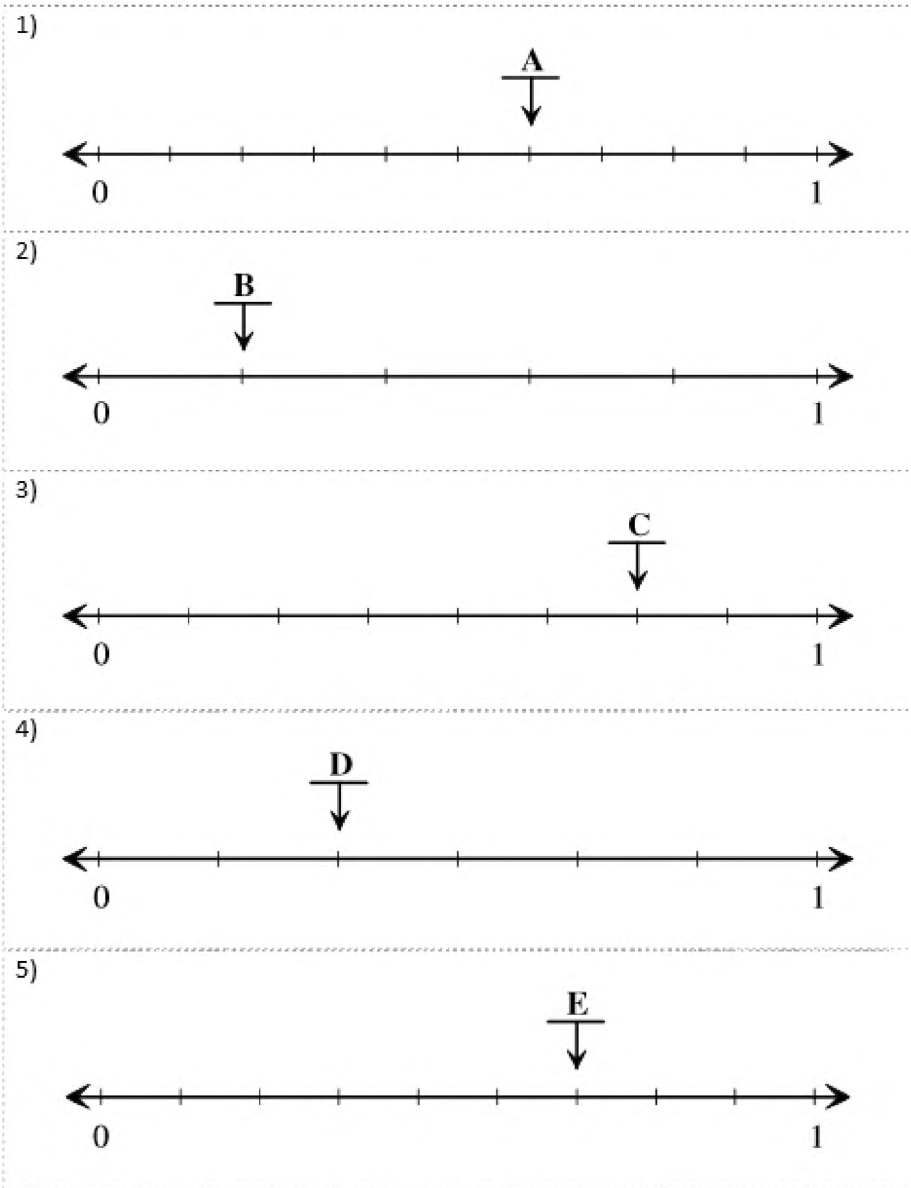
Name:

Worksheet 3: Identify fractions on a number line

In this activity learners should be able to identify a fraction on a number line. Learners should develop an understanding that fractions are numbers with unique positions on a number line. What fraction do the letter points to?



Answers

Worksheet adapted from http://www.mathworksheets4kids.com

a) Describe the process of getting the answers.

……………………………………………………………………………………………

……………………………………………………………………………………………

b) What are the common properties for all fractions you have written above?

……………………………………………………………………………………………

……………………………………………………………………………………………

Name:

Worksheet 4: Locating fractions on a number line

This activity is a continuation of activity 3. In this activity, learners should be able to draw a number line and indicate the position of a given fraction.

The teacher will use the first problem to show you how to complete the activity.

1. Use a ruler and pencil to draw a number line to locate each given fraction.

a) $\frac{5}{8}$

b) $\frac{3}{4}$

c) $\frac{8}{7}$

d) $\frac{7}{4}$

e) $\frac{12}{3}$

f) Explain the process of using a number line to write $\frac{19}{5}$ as a mixed number.

....

....

....

....

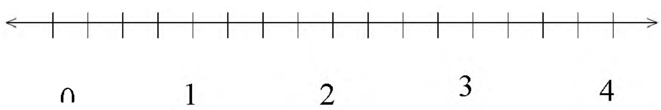
....

Name:

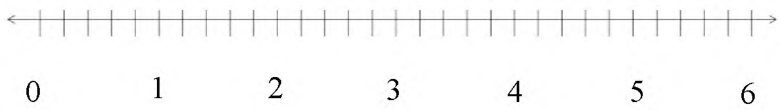
Worksheet 5: Locating fractions on a number line

Place each fraction on a number line. Use a pencil to mark the position of the given fraction. Explain briefly what you did to get the position of each fraction.

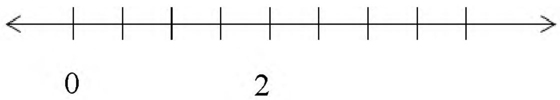
a) $\frac{3}{4}$



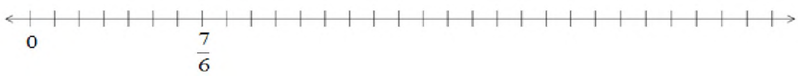
b) $2\frac{3}{5}$



c) $\frac{3}{2}$



d) $2\frac{2}{6}$

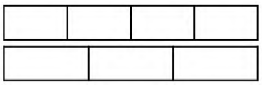


Name:

Worksheet 6A: Comparing fractions

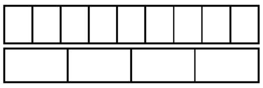
Compare the fraction on the left to the fraction on the right using the symbols, greater than (>), less than (<), or equal to (=). Insert the correct symbol to make each statement true. You can color the fraction bars to help you.

a)



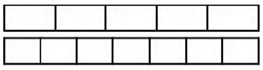
$\frac{1}{4}$ $\qquad$ $\frac{1}{3}$

b)



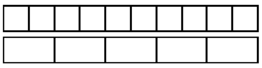
$\frac{3}{4}$ $\qquad$ $\frac{3}{9}$

c)



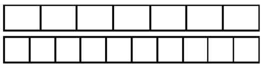
$\frac{2}{7}$ $\qquad$ $\frac{2}{5}$

d)



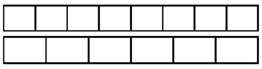
$\frac{4}{5}$ $\qquad$ $\frac{4}{10}$

e)



$\frac{1}{10}$ $\qquad$ $\frac{1}{7}$

f)



$\frac{5}{6}$ $\qquad$ $\frac{5}{8}$

What did you find special about comparing these fractions?

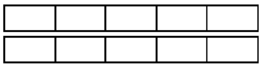
....

Name:

Worksheet 6B:

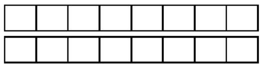
Compare the fraction on the left to the fraction on the right using the symbols, greater than (>), less than (<), or equal to (=). Insert the correct symbol to make each statement true. You can color the fraction bars to help you.

a)



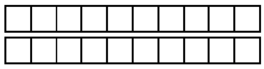
$\frac{2}{5}$ $\frac{1}{5}$

b)



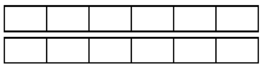
$\frac{5}{8}$ $\frac{6}{8}$

c)



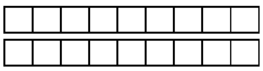
$\frac{3}{10}$ $\frac{7}{10}$

d)



$\frac{4}{6}$ $\frac{5}{6}$

e)



$\frac{7}{9}$ $\frac{2}{9}$

What did you find special about comparing these fractions?

....

How do you compare one fraction to another fraction? Create fraction comparisons of your own in the boxes below. In box #1, show that the fraction on the left is LESS THAN (<) than the fraction on the right. In box #2, show that the fraction on the left is GREATER THAN (>) the fraction on the right. Using what you have learned in Part A and B, explain why each comparison in the boxes is correct.

....
....
....
....
....

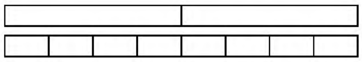
Box #1   <
Box #2   >

Name:

Worksheet 6C:

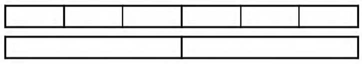
Compare the fraction on the left to the fraction on the right using the symbols, greater than (>), less than (<), or equal to (=). Insert the correct symbol to make each statement true. You can color the fraction bars to help you.

a)



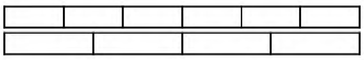
$\frac{1}{2}$ $\frac{3}{8}$

b)



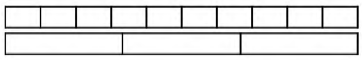
$\frac{2}{2}$ $\frac{6}{6}$

c)



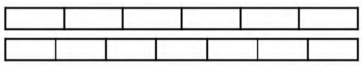
$\frac{1}{4}$ $\frac{2}{6}$

d)



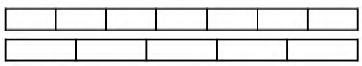
$\frac{2}{3}$ $\frac{5}{10}$

e)



$\frac{6}{7}$ $\frac{5}{6}$

f)



$\frac{4}{5}$ $\frac{3}{7}$

What did you find special about comparing these fractions?

....

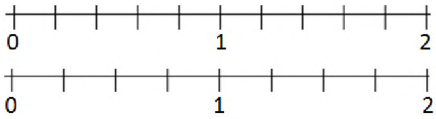
....

Name:

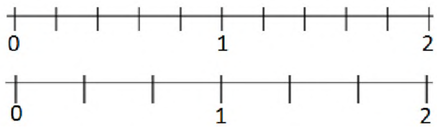
Worksheet 7: Comparing fractions using number lines

1. Use the following number lines to determine which fraction is larger?

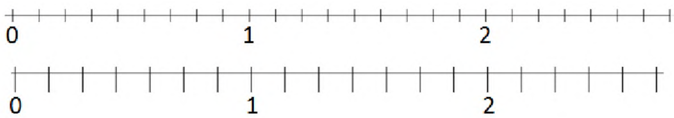
a) $\frac{5}{4}$ $\qquad \frac{6}{5}$



b) $\frac{4}{5}$ $\qquad \frac{2}{3}$



c) $\frac{5}{9}$ $\qquad \frac{4}{7}$



2. Draw the number lines to compare the fractions

a) $\frac{5}{12}$ $\qquad \frac{4}{10}$

b) $\frac{7}{5}$ $\qquad \frac{9}{8}$

c) Explain how you use a liner number to compare $\frac{7}{5}$ and $\frac{9}{8}$.

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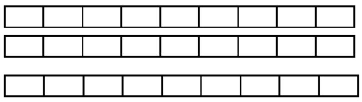
……

Name:

Worksheet 8A: Ordering fractions

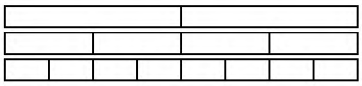
List the following fractions in order of size. You may shade the fraction bars to help you.

a)



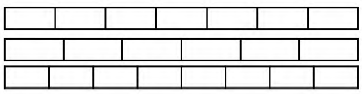
$\frac{5}{9}$; $\frac{2}{9}$; $\frac{6}{9}$

b)



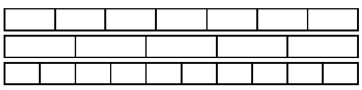
$\frac{1}{2}$; $\frac{3}{4}$; $\frac{2}{8}$

c)



$\frac{5}{7}$; $\frac{5}{6}$; $\frac{5}{8}$

d)



$\frac{6}{7}$; $\frac{4}{10}$; $\frac{3}{5}$

Name:

Worksheet 8B: Ordering fractions

Draw a number line to locate each fraction. Then, use your number lines to order the given fractions in ascending order.

a) $\frac{4}{8}; \frac{4}{10}; \frac{4}{5}$

b) $\frac{1}{4}; \frac{2}{3}; \frac{4}{10}$

c) $\frac{5}{7}; \frac{1}{3}; \frac{2}{8}$

Name:

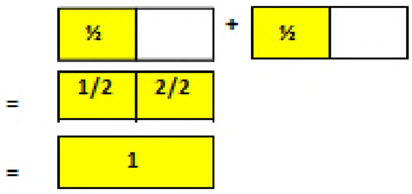
Worksheet 9: Adding fractions

This activity is designed to help learners to understand addition of fractions using common denominators.

Calculate the sum of each fraction. Then, use circles or fraction bars to show the process of getting the sum of fractions.

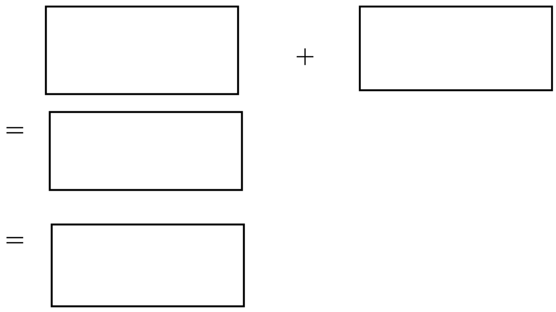
a)

$$\frac{1}{2}+\frac{1}{2}=\frac{2}{2}=1$$



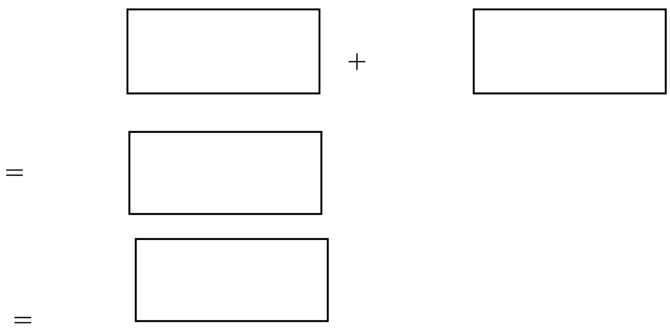
b)

$$\frac{2}{3}+\frac{1}{6}$$



c)

$$\frac{1}{2}+\frac{1}{3}$$



d)

$$\frac{4}{5}+\frac{3}{7}$$

$\boxed{} \quad + \quad \boxed{}$

$= \boxed{}$

$= \boxed{}$

APPENDIX B: PRE-TEST

Learner's name: .... Date: ....

Mathematics Pre-test: Common Fractions **Duration: 45minutes**

Instructions to learners:

- Write your name and date in the space provided.
- Answer all questions.
- Show all your workings.
- The use of calculators is not allowed for this test.

---

1. Naming and Identifying fractions

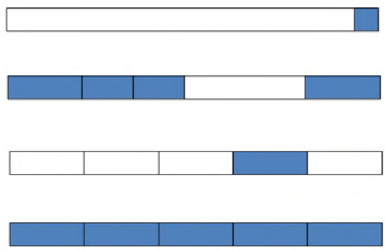
1.1. Determine which shaded areas represent $\frac{1}{5}$ of the bar.

A

B

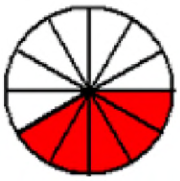
C

D



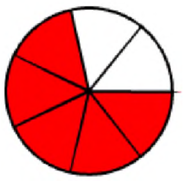
Answer: ....

1.2. What fraction of the circle is the shaded area?



Answer: ....

1.3. Determine which letters best describe the shaded parts of the whole circle.



A Five sevenths
B Two sevenths
C Five quarters
D Two quarters of seven

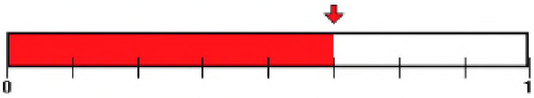
Answer: ....

1.4. A fraction $\frac{3}{4}$ reads 'three quarters'.

Write the fraction $\frac{5}{12}$ in words.

Answer: ....

1.5.Below is a number line with an interval from zero to one.
What fraction is shown by the shaded length on the number line?

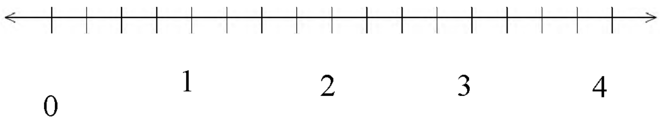


Answer: ....

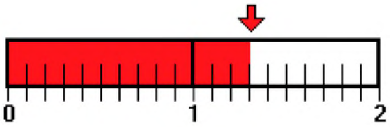
1.6.What fraction does the letter D point to?

Answer: ....

1.7.Mark the position of the fraction $\frac{3}{4}$ on the number line with cross (**x**).

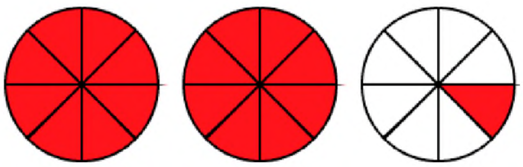


1.8.What fraction is shown by the shaded length on the number line?



Answer: ....

1.9.The circles below represent a fraction. One circle represents a whole number.



1.9.1. What fraction do the shaded areas (all together) represent?
Write your answer as an improper fraction.

Answer: ....

1.9.2. Write your answer as a mixed number.

Answer: ....

1.9.3. Draw this fraction on a number line.

2. Comparing fractions

2.1.Compare the fractions and fill in the space with one of the symbols <, > or =to make each statement true.
Explain in words or use the drawing to show how you got your answer.

2.1.1. $\frac{2}{7}$ $\qquad\qquad$ $\frac{3}{7}$

2.1.2. $\frac{4}{5}$ $\frac{4}{6}$

2.1.3. $\frac{1}{2}$ $\frac{3}{8}$

2.1.4. $\frac{4}{12}$ $\frac{1}{3}$

3. Ordering fractions

3.1.List the following fractions in order of size, starting with the smallest.
Please show how you worked out each answer. You may use a drawing to help you to explain.

3.1.1. $\frac{1}{2};\frac{1}{4};\frac{1}{3}$

3.1.2. $\frac{8}{10};\frac{3}{10};\frac{5}{10}$

3.1.3. $\frac{2}{3};\frac{1}{12};\frac{5}{5}$

4. Adding fractions

4.1.Calculate the sum. Explain how you got the answer. You may use a drawing to help you to explain the process of getting the sum of the fractions.

4.1.1. $\frac{1}{8}+\frac{3}{8}=$

4.1.2. $\frac{3}{4}+\frac{4}{5}=$

Thank you.

The End!

APPENDIX C: POST-TEST

Learner's name: .... Date: ....

Mathematics Post-test: Common Fractions **Duration: 45minutes**

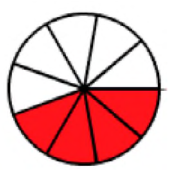
Instructions to learners:

- Write your name and date in the space provided.
- Answer all questions.
- Show all your workings.
- The use of calculators is not allowed for this test.

---

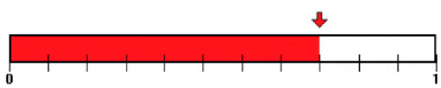
1. Naming and Identifying fractions

1.1 Write in words the fraction of the shaded area of a circle.



Answer: **...**

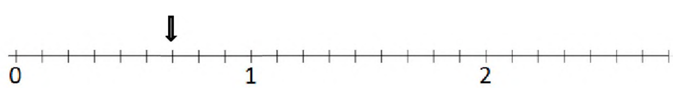
1.2 Determine which letters best describe the shaded length of a number line.



A Eight eleventh
B Eighth elevenths
C Eight elevenths
D Eighth eleventh

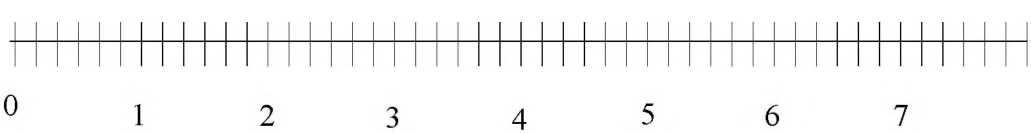
Answer: **.................................**

1.3 Identify the fraction at an arrow on a number line.

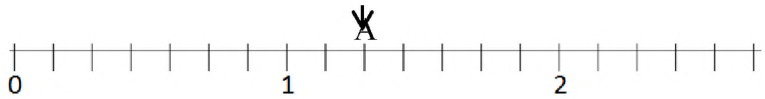


Answer: **.................................**

1.4 Mark the position of the fraction $\frac{5}{6}$ on the number line with cross (**x**).

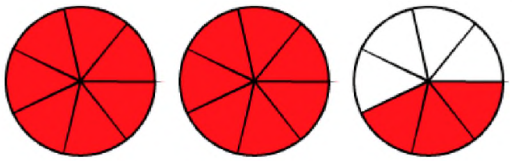


1.5 What fraction does the letter ***A*** point to?



Answer: ....

1.6 The circles below represent a fraction. One circle represents a whole number.



1.6.1 What fraction do the shaded areas (all together) represent?
Write your answer as an improper fraction.

Answer: ....

1.6.2 Write your answer as a mixed number.

Answer: ....

1.6.3 Draw this fraction on a number line.

2. Comparing fractions

2.1 Compare the fractions and fill in the space with one of the symbols <, > or =to make each statement true.
Explain in words or use the drawing to show how you got your answer.

2.1.1 $\frac{5}{7}$ $\qquad$ $\frac{5}{10}$

2.1.2 $\frac{4}{8}$ $\qquad$ $\frac{3}{8}$

2.2 Draw the number lines to compare these fractions. Explain how you obtained your answer using the number lines.

$\frac{9}{8}$ $\qquad$ $\frac{7}{5}$

3. Ordering fractions

3.1 List the following fractions in order of size, starting with the smallest.
Explain how you worked out the answer. You may use a drawing to help you to explain.

$\frac{3}{10};\frac{3}{4};\frac{3}{8}$

3.2 Draw a number line to locate each fraction. Then, use your number lines to order the given fractions in ascending order. Explain also how you obtained your answer.

$\frac{2}{5};\frac{1}{12};\frac{4}{9}$

4. Adding fractions

4.1 Calculate the sum. Explain how you got the answer. You may use a drawing to help you to explain the process of getting the sum of the fractions.

4.1.1 $\frac{1}{5}+\frac{3}{5}=$

4.2 Calculate the sum of each fraction. Then, use the fraction bars to show the process of getting the sum of fractions.

4.2.1 $\frac{5}{8}+\frac{3}{4}=$

+ =

4.2.2 $\frac{2}{3}+\frac{2}{5}=$

+ =

Thank you. **The End!**

APPENDIX D: PRE-INTERVIEW

Pre-interview questions
Greet the interviewee and welcome him/her to the interview.
Assure the confidentiality and unanimity of the interviewee.

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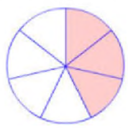
1. **Naming and identifying of fractions**



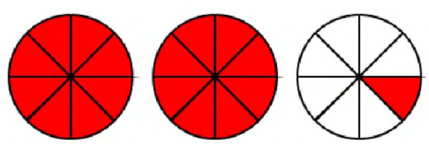
a) In the pie above, what fraction of the circle is shaded?
Explain the process of how you obtained the fraction.

b) (i) Please draw a number line and put two thirds on it.

(ii) Please mark five thirds and label it for me.

c) (i) Which mixed number is equal to $\frac{13}{6}$?

(ii) Ask the learner to draw a number line to illustrate your answer.

d) 

(i) What fraction do the shaded areas (all together) represent?
Write your answer as an improper fraction.

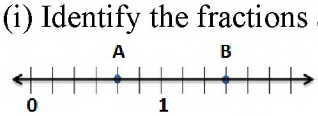
Answer: **................................**

(ii) Write your answer as a mixed number.

Answer: **................................**

(iii) Draw this fraction on a number line.

e) (i) Identify the fractions at points A and B on the number line:



(ii) Explain how the answer at A is different from the answer at B?

2. Comparing fractions
Which of the two fractions is larger: 2/3 or 3/10?

3. Ordering fractions

a) List the following fractions in order of size, from the smallest to the largest?

$\frac{4}{5}; \frac{1}{9}; \frac{2}{7}$

b) Use circles or rectangular bars to show the way to get the answer.

4. Adding fractions

a) Calculate the sum $\frac{2}{5} + \frac{1}{6}$.

b) Please illustrate your answer using fraction bars.

Thank you.

The end

APPENDIX E: POST-INTERVIEW

Post-interview questions **Learner's name: ....**

Greet the interviewee and welcome him/her to the interview.
Assure the confidentiality and unanimity of the interviewee.

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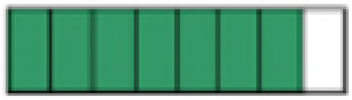
1. **Naming and identifying of fractions**



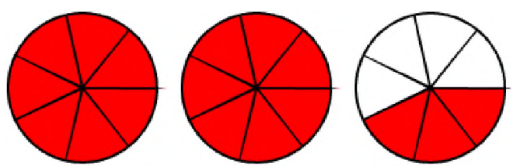
a) In the pie above, what fraction of the circle is shaded?
Explain the process of how you obtained the fraction.

Answer: **.............................**

b) (i) Please draw a number line and put four sevenths thirds on it.

(ii) Please mark 14 sevenths on the same number line.

c)



(i) What fraction do the shaded areas (all together) represent?

Write your answer as an improper fraction.

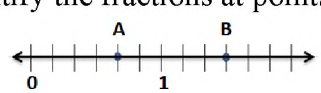
Answer: **................................**

(ii) Write your answer as a mixed number.

Answer: **................................**

(iii) Draw this fraction on a number line.

d) (i) Identify the fractions at points A and B on the number line:



(ii) Explain how the answer at A is different from the answer at B?

2. Comparing fractions

Which of the two fractions is larger: 5/8 or 3/10?

3. Ordering fractions

a) List the following fractions in order of size, from the smallest to the largest?

$\frac{3}{5};\frac{4}{9};\frac{2}{7}$

b) Use fraction bars or number lines to show the process of getting to the answer.

4. Adding fractions

a) Calculate the sum $\frac{2}{5}+\frac{2}{3}$.

b) Please illustrate your answer using fraction bars.

Thank you.

APPENDIX F: TRANSCRIPTION OF PRE-INTERVIEWS

T: The first question I want you to look at the pie that you, the pie chart that you see here. Write for me for me the fraction that is shown by the shaded area.
L3: It is three over seven.
L3: One over four. The correct answer is 4/6.
T: One out of four, how did you that?
L3: The pieces are four.
T3: Which pieces, pieces where? Which pieces?
L3: These sticks between zero and one.
T: Okay. Can you count them?
L3: They are four.
T: Uhm.
L3: One, two, three, four. Until A here.
T: Okay. So, they are four?
L3: Yes.
T: Out of how many?
L3: Only one.
T: I want you to look at this circle here. It is divided into parts and part of it is shaded. Can you tell me what part of the fraction on circle is shaded?
L5: Proper fraction.
T: No, what part of the fraction that is shaded, the part that is shaded, what fraction do you see? Can you write it down here [at the answer space]?
L5: Three out of seven.
T: Explain how you find the fraction of the shaded part of a circle.
L6: For you to find the numerator you have to count the shaded parts, so they are one two three, that is a numerator. And then for the denominator, you count all the parts that the pie is divided into. So they are seven in total and the denominator would be seven. See 00:00-0:22.
T: Look at the pie here, at this circle representing a pie, now what fraction do you see there of the shaded part?
L7: It is three over seven... three seventh ... three is the number of shaded parts and seven is the number of equal parts the whole is divided into.
T: Okay. So one represents what? This one represents?
L3: The whole.
T: Four represents?
L3: The pieces.
T: Okay, how did you get that answer?
L3: I got it by looking at how many parts are shaded and the whole thing is cut into seven pieces.
T: Uhm.
L3: So, I got my answer from the three parts that are shaded and together the pieces are seven. And the whole thing is just one whole number, and the circle.
T: So, that fraction represents what? It is equal to what? The fraction that you wrote it shows what? L3: It shows the shaded part.
T: The shaded part of what?
L3: Of the whole.
T: Okay. Thank you very much. Can you please help me to draw this fraction on a number line here?

L3: Okay. L3 used a pencil and ruler to draw a number line and mark the position of a given fraction 2/3. Firstly, L3 Located 2/3 at 2 by shading length of 2 whole on a number line.
T: It is enough. It is enough. So, where is two third?
L3: Two third is here [pointing at 2 on the number line]. The whole thing ends here at three. And then two is from here until here [shifted his finger from 0 up to 2].
T: Okay, where is one?
L3: One is here.
T: Okay. Can you write for me two third as a fraction? L3 wrote 2/3 correctly.
T: Okay. Let's look at that fraction. Is that fraction greater than 1 or less than 1?
L3: No, it is less than one.
T: Okay. Now, look at what you shaded. Is that greater or less than 1? What you shaded on the number line.
L3: It is greater than 1.
T: Okay. Uhm, can you try now to show that 2/3 is less than 1? L3 erased the first shading and shaded the area less than one showing that 2/3 is less than one.
L3: It is less than 1.
T: How do you know that two third is here?
L3: Because from here the ticks are two. And here the whole line is up to three.
T: Hmm.
L3: So the ticks are two mos.
T: Okay.
L3: It is like pieces, so is two over three.
T: Okay. Let's move on. Now. Can you move to part (b)? Part (b) asks you to find the mixed number that is equal to 13 over 6. Can you write 13 over six as a mixed number? The teacher decided to read the fraction 13/6 as "13 over 6" to make communication feasible, as this is the way L3 read fractions. L3 recorded 1 2/6 for 13/6.
T: Okay, hmm. Can you show me that on a number line or can you draw that on a number line? Or can use pictures to show that this mixed number is equal to this or this improper fraction thirteen over six is equal to 1 and two over six? Or how do you know that these two are equal? What did you do?
L3: I divided thirteen into six.
T: Then?
L3: I divided six from thirteen. Then I got that six goes into thirteen, two times and then there is one which is a reminder. This time L3 recorded a correct mixed fraction 2 1/6.
T: Okay.
L3: So one is going to be whole and the equal two for how many six goes into thirteen, and it will be the numerator and over the denominator and the denominator does not change. L3 exchanged the whole and numerator of the mixed number in this explanation, making the explanation incorrect and not correct as the previous answer. L3 did not use the picture in fig 2 to explain this time.
T: Okay. Which one is a whole? How many times does … show me the number that shows how many times six goes into thirteen. Which number?
L3: This two [pointing at two of 2 1/6].
T: And the remainder you put it where?
L3: Here [pointing at one of 2 1/6].
T: Okay. Thank you very much. Can we now move onto ... Let's move onto this part, the next page? Ne? Uhm, question (d)? I want you to find, to look at this, ne? This circle this represents a whole, ne? Yes. And now these whole are divided into equal parts. If you consider this as one

picture, so it represents an amount of something, represented as a fraction. Ne? Okay. Now, what fraction by these three circles? Considering the shaded parts?

L3: The shaded parts?

T: Uhm, you have three circles but you write the shaded parts as a fraction. And I want you to write your answer as an improper fraction.

L3: Together they are seventeen, the shaded parts.

T: Uhm. So write your answer here [in the answer space provided on the interview sheet].

L3: And the pieces are eight.

T: Okay. Now, can you write that number seventeen over eight as a mixed number? L3 recorded 1 2/8.

T: Okay, how did you get that mixed number?

L3: I divided seventeen with eight.

T: Then?

L3: So, eight goes into seventeen two times.

T: Then?

L3: And then the remainder is one.

T: Okay. Thank you very much.

L3: One is a whole number, two is a numerator and eight is a denominator. See 09:59-10:04. T: So, the numerator here represents? Two represents?

L3: How many times eight goes into seventeen.

T: Okay. Then, we move onto ... this question asks you to draw this fraction on a number line but we skip it and then we move onto question (e).

L3: Okay.

T: Can you identify the fraction at position A? You see where is A?

L3: Yes.

T: Okay. What fraction is there? You can just write down your answer. What fraction do you think is there? L3 recorded A=1/4 and B=4/1……………………………………………….. n a number li e

T: Okay, I want you now to look at this question (b) (i).

L5: Okay.

T: I want you to draw a number line here.

L5: Okay.

T: You can use a pencil and then I want you to put two thirds on that number line.

L5: Okay. L5 indicated the fraction two third by shading the distance between 2 and 3 on the number line.

T: We move to this question. Which mixed number is equal to thirteen sixths?

L5: Six.

T: Can you just write that for me? L5 recorded 6/13 as mixed fraction for 13/6.

T: Let's move onto the next question. This question is having three circles. But one circle represents one whole number.

L5: Okay.

T: Can you write the fraction shown by the shaded area? And I want you to write your answer as an improper fraction. L5 recorded 1/24.

T: Write your answer of this fraction as a mixed number. L5 recorded 24/1 as a mixed fraction for 1/24.

T: What fraction is at point A and B on a number line?

L5: The fraction?

T: Yes.

L5: It is an improper fraction.
T: Write that down. L5 recorded A=1/4 and B=4/1. See 09:27 – 10:20.
T: Okay, can you draw this or show this fraction 2/3 on a number line?
L6: Actually, we were not taught about this, but then I do want to try.
T: Can you try that?
L6: Yes
T: Okay. L6 shaded up to 2 on the number line of 0-3 interval.
T: What number is here?
L6: It is two.
T: You have shown the length of two, ne?
L6: Yes.
T: Do you think this can be equal to two over three?
L6: I think so.
T: Okay. Now if I ask you to say where is two here, if I just say show two things? If you have two things?
L6: Okay.
T: Why are you saying two is the same as two over three?
L6: I am saying, this because in the circle, two parts are shaded and the third part is not shaded.
T: Okay, how much is one part of a circle?
L6: I think is one centimetre. T: No, I meant how much is one shaded part? L6: I think. L6: Yes.
T: Okay, the next question, this number line is divided ... it has a distance from zero to one, is divided into three equal parts. From one to two it has the same number of parts. Can you place for me six thirds? Can you write for me six thirds? L6 located 6/3 at 2 and 10/6 at 3 on her drawn number line.
T: Can you write for me six thirds as a symbol? Can you label for me ten sixths?
T: Which mixed fraction is equal to thirteen sixths? Can you just write for me thirteen over six?
L6 wrote down 13/6. L6: For you to get a mixed number, and since this is improper fraction and the numerator is bigger than the denominator and this line is called the division line.
T: Okay.
L6: You have to know how many times six goes into thirteen so it goes in 2 times and a remainder of one that will become a numerator and then the denominator does not change it remains six.
T: Okay, thank you very much. Look at this number line here. What fraction do you think is at A? L6: I think it is five over thirteen.
T: How did you get that answer?
L6: I start counting from zero until at A.
T: So you counted... How?
L6: One two three four, sorry, it is four not five. So is four over thirteen.
T: How did you get thirteen?
L6: I counted from here till here. L6 recorded A=4/13 [instead of 4/6].
T: What if I add another line there? Then it will be what?
L6: Four over fourteen.
T: Okay. What fraction is at B?
L6: Nine over thirteen [the fraction is 9/6].
T: Okay. Thank you, let's look at question (b) and i want you to draw a number line and put two third on it. L7 first located 2/3 at two, and then located 2/3 appropriately after probing by the teacher.
T: What number is here?

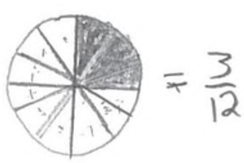
L7: Two.
T: Now what number is two thirds, is it a whole number or a fraction?
L7: It is a fraction.
T: Can you say two is equal to two thirds?
L7: ... No [shaking his head]. Come on again?
T: You see I wanted you to mark where two third but you have here is two?
L7: Ooo!
T: Now where is one?
L7: One is here.
T: On the same number line, can you mark for me five thirds? L7 recorded that 13/6=2 1/6.
T: One circle represents one whole number. I want you to look at these three circles and tell me what fraction is represented by the shaded part of the circles? Write your answer as an improper fraction. L7 recorded 17/8 and 2 1/8 for questions (d) (i) and (d) (ii). See 09:08-09:53.
T: Let's move to question (e). I want you to tell me what fraction is at point A and point B? L7 recorded that A=4/6 and B=1 3/6. See 10:42-11:07.
T: Okay. Thank you. We move onto ... We are about to finish now. Can we move to question 2? Which of these two fractions is bigger? Which one is larger, between 2/3 and 3/10, which one is larger?
L3: 2/3 is bigger than 3/10. L3 used LCD method to compare fractions.
T: Why are you saying that? How do you know that it is bigger?
L3: First you have to look for the same lowest common denominator, for both the denominators. When you get them, and then you multiply like a times like this. You multiply the denominator on this side and this denominator on this side. And then you get the same LCD.
T: Then?
L3: And for you to be able to compare.
T: Then?
L3: These numerators, make sure you can see which one is small. If the denominators are the same, you will be able to identify which one is big.
T: So, now which one is big?
L3: 2/3 is bigger than 3/10.
T: So, using the new fraction you have here, why are you saying 20/30 and 9/30?
L3: Because 20 is bigger than 9.
T: Then? Okay. And then we come here. One of the learner in your class said one over four or a quarter is less than one seventh. Do you agree?
L3: Is not bigger.
T: So what is the right answer? Which one is bigger there?
L3: One over four or a quarter.
T: So, okay. So how did you work out that one?
L3: It is the same as there [refereeing to her answer in question 2. (a) above].
L3: You have to look for the same denominator. And here like, if it is four pieces.
T: Then?
L3: And it is an apple and then you got it in four pieces and here this apple and you cut it into seven pieces, the one cut into four pieces would be bigger than the one cut into seven pieces. But is just one apple.
T: Okay, now, what does one over four represents in terms of the size of the apple? How many pieces of an apple is one over four?

L3: Four pieces. T: No, one over four represents how many pieces when you cut the apple into four. How many pieces are represented by one over four? The learner could not use one over four to count the number of pieces of an apple she refers to.
L3: I can’t get it.
T: You know, you said if you cut the apple into four equal pieces. And then now, I am asking one over four represents how many pieces of an apple?
L3: Two.
T: Two? How do you know? Can you show me that, just there [referring to an interview worksheet]. L3 drew a partitioned circle with 4 equal parts and shaded a quarter of the circle, then she got the correct answer.
L3: Just one piece.
T: One piece? Are you sure?
L3: Yes.
T: Okay. One piece, out of how many?
L3: Four.
T: Okay. Then, our last part is now here whereby we have to look at ordering. Can you order for me four fifth, one ninth and two seventh? Order them from the smallest to the biggest. L3 recorded 4/5<2/7<2/9.
T: Can you just explain how did you find the answer?
L3: I checked that how many pieces they are.
T: Then?
L3: So one piece over nine.
T: Then?
L3: They are small just.
T: Then?
L3: Just one piece. And the second one, they are two pieces over seven.
T: Then?
L3: And then the third one there are four pieces over five.
T: Now, why did you say one over nine is the smallest? L3: Because is just one piece. T: Is that a piece or a fraction?
L3: It is a fraction.
T: Okay.
L3: Like a fraction like this and then you cut into nine pieces.
T: Which of those fractions is greater than 1?
L3: Greater than 1?
T: Yes!
L3: Is four.
T: Which one of those pieces is greater than one as a whole?
L3: One whole?
T: Then?
L3: They are greater than one.
T: Which of those fractions is greater than 1? How many of those fractions are greater than 1?
L3: No. Is none.
T: Okay, of those fractions is greater than a half there? A half of a whole?
L3: No, there is none.
T: You know a half right?
L3: Yes, I know a half.
T: And, so they are all smaller than a half?

L3: Yes.
T: A half of a whole?
L3: Now, I understand. L3 erased the first answer which was correct and wrote an incorrect answer.
T: Why did you rearrange it?
L3: Because I realized that the whole piece is bigger.
T: Which one is bigger?
L3: One out of nine.
T: Then?
L3: The whole piece is bigger than all the other pieces.
T: Let's continue. Between these two fractions [2/3 and 3/10], I want you to compare these two fractions and tell me which one is bigger?
L5: Three out of ten is bigger than two out of three.
T: Okay, how do you know? How do you know that this one is bigger than the other one?
L5: Because, two is smaller than three. The denominator ten is bigger than three.
T: Then?
L5: And then it became three out of ten.
T: Then?
L5: The answer will be ...
T: Bigger.
L5: Bigger than two out of three. See 12:02-12:30.
T: Okay. Let's look at this one. One of your classmate was comparing the two fractions. One quarter and one seventh. And then his conclusion says one over seven is greater than one quarter. Do you agree?
L5: Yes. See 12:48-13:17.
T: How do you know that it is true?
L5: Because, the numerators are the same
T: Then?
L5: And the denominators are not the same.
T: So?
L5: Because four is smaller than seven.
T: Then?
L5: It becomes one out of seven.
T: so one out of seven is what? Is it smaller or the bigger than?
L5: It is the bigger one.
T: Okay. Thank you very much. And then, can you order for me these fractions, four fifths, one ninth and two sevenths in ascending order, from the smallest to the biggest? You can write your answer down there. L5 recorded 1/9<2/7<4/5.
T: How do you know that one over nine is the smallest?
L5: Sir, you said in ascending order?
T: Yes, from the smallest fraction to the biggest one.
L5: Ooo! This one is wrong sir.
T: So it is wrong?
L5: Yes.
T: Okay. L5 erased his first ordering and recorded 4/5<2/7<1/9.
T: So, the smallest fraction there is what?
L5: Is four over five.
T: Okay, how do you know that it is the smallest?

L5: Because the numerator is than... Two. Is smaller than other numbers.
T: Which are?
L5: Two! They are bigger numbers from two out of one.
T: Which bigger numbers are those?
L5: The biggest number is four.
T: Okay.
L5: The numerator is smaller than nine.
T: Okay.
L5: Now, the smallest number is four out of five.
T: Okay.
L5: And then two out of seven and then one out of nine.
T: Okay.
L5: And now the big number is one out of nine.
T: Why is one out of nine the biggest?
L5: Because, the numerator is smaller than other numbers.
T: Then?
L5: But the denominator is bigger than.
T: Okay, so, are you saying we should only look at the how big the denominator is?
L5: Yes.
T: Okay.
L5: Yes.
T: That is why you are saying the smallest one with the smallest fraction should be the first one?
L5: Yes. See 14:49-16:21.
T: Which of these two fractions 5/12 and 5/8, do you think is larger?
L6: is 5/8.
T: Why do you say 5/8 is bigger?
L6: It is a long process.
T: Okay.
L6: If the denominators are not the same so you have to change them. You either look for the lowest common denominator ... and by that you have to ... the first fraction you take the denominator and then you multiply both the numerator and denominator. And on the other side you take the first denominator and then you multiply both numerator and denominator. If for example, there is two over four and six over eight, so you know that there is a number that can multiple with four to give you eight and that is two so you do not need to find the LCD you just have to multiply one fraction so it will give you the same denominator, so you compare.
T: One of your classmates gave an answer that one over four is smaller than one over seven. Do you agree?
L6: No, one over seven is less than one over four. For example if four people are sharing the same bread that other seven people will share among themselves, those who are seven will get smaller pieces than the four who are sharing. So, one over seven is smaller.
T: Thank you. Can you order these fractions one fifth, one third and a quarter in ascending order? L6: the answer is one over five then one over four and one over three.
T: How do you?
L6: Just like in the previous example, and if you use diagrams one over five will be smallest and one over three goanna be the biggest.
T: Let’s look to comparing fractions. Between these two fractions 2/3 and 3/10. Which one is bigger?
L7: It is two thirds.

T: How do you know?
L7: because it is closer to a half than three tenths.
T: Thank you very much. Let's look at these other three fractions, and order them from the smallest to the biggest? L7 recorded 1/9<2/7<4/5.
T: How do you know that this ordering is correct?
L7: I checked which one closer to a half or one.
T: Okay.
L7: If that does not work, you change the denominators so that they can all have the same denominators and then you can compare the numerators because if they are the same then these quantities should tell you how big is the value. See 11:56-12:22.
T: Okay. Can you calculate for me the sum of two fifths and one sixth? L3 used the LCD method to add fractions. The answer was correct.
T: How did you work out the answer?
L3: I found out the LCD for five and six, and then also the numerator.
T: Then?
L3: And then i plus the numerator to the numerator. L3 explained that it is important to find the LCD of the denominators first so that once the denominator is the same, then you can add the fraction. When, I asked why she can't just add fractions without changing the denominator, she explains that she have to, because that's the way she was taught. See 24:20-25:00.
T: Can you please draw diagrams to show that two fifths plus one sixth can gives you a sum of seventeen thirtieths? Are able to draw that for me, using circle or fraction bar?
L3: How? L3 tried to draw fraction bars using the examples given by the teacher. However, L3 did not use the fraction bars to get the correct sum. Her answer became three elevenths.
T: Okay, can you calculate for me now the sum of these two fractions two fifths plus one sixth?
L5: is three out of twelve.
T: Okay. Just write it for me there. L5 recorded 2/5+1/6=3/12.
T: Three out of? How do you get twelve?
L5: Because five plus six gives you twelve.
T: Okay. L5: and two plus one gives you three. See 16:44-17:03. T: Okay. Can you use rectangular bars to show that two fifths plus one sixth is going to give you three twelfth? Can you do that?
L5: Yes.



T: Now we are moving to the last part. Calculate the sum of two over five plus one over six?
L6: In mathematics there is a rule that says you do not add or subtract fractions when the denominators are not the same, so you still have to look for the lowest common denominator. Which is 30, and the final answer is seventeen over thirty. See 14:14-15:25. L6 added that she find comparing of fractions difficult sometimes especially when she is given four fractions to compare and their denominators and numerators are different because they were not taught how to find the LCD for comparing. See 16:28-17:12. L7 used the LCD to add fractions correctly.
T: Use rectangular bars to show the process of adding fractions? L7 said that he had never used the diagrams to add fractions although they have done conversion of fractions to percentages in grade 7 to compare fractions. L7 added that he found the pre-test and pre-interview questions easy.

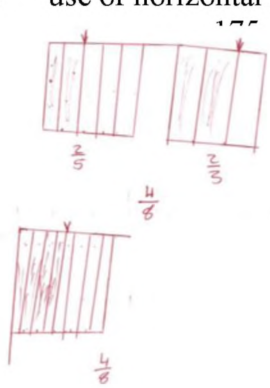
APPENDIX G: TRANSCRIPTION OF POST-INTERVIEWS

T: Can you please tell me the fraction shown by the shaded area of this fraction bar?
L3: It is seven eighths.
T: I want you to tell me what fraction is shown by the shaded area of the fraction bar. L5: Seven eighths.
T: What does four and seven represents. L5: Four represents... seven represents a whole. T: Hmm. L5: Four represents shaded parts. See 02:31-02:55v3.
T: Here you have a fraction bar and part of it is shaded. What part of the fraction bar is shaded?
L6: It is seven eighths. See 00:26-00:36.
T: What does seven and eighths represents
L5: Seven represents shaded parts and eight represents pieces of eight, eight represents in pieces.
T: Hmm.
L5: The size of pieces they are eight.
T: The size of? ... Of the pieces.
T: Where?
L5: They are equal.
T: Those pieces of a what?
L5: Of an eight of, a bread like that. See 00:19-01:06 v2.
T: What does seven means?
L6: Sevens represents the shaded parts. And eight represents the parts that are divided in by one. 7/8 is a proper fraction because the numerator is less than the denominator. See 00:28-00:38 and 00:43-00:55, 01:30-01:40.
T: I want you to use a ruler to draw a number line and indicate for me four sevenths on that number line? L3 located 4/7 correctly, and partitioned the number line appropriately based on the fraction denominator. L3 located and identified fractions greater than one whole unit using both circles and the number line. See 04:19-04:57 and 06:52-07:22.
T: I want you tell me what fraction is at point A and point B on the number line? L3 recorded the fractions at A and B as 4/6 and 9/6, which are the correct fractions.
T: What does each fraction at A and B means? L3 described the difference between fractions at A and B as proper and improper fractions. She also defined each type of fractions correctly. She added that that an improper fraction is a fraction greater than 1. See 11:17-12:21.
T: I want you to draw a number line there, using a pencil and indicate four sevenths on that number line. L5 located 4/7 on the number line appropriately.
T: How do you that the answer is there?
L5: Improper fraction.
T: No, how do you know that the answer will be there?
L5: Because I draw a number line.
T: Okay!
L5: And then we put ... we start at zero.
T: Hmm.
L5: And then the first column will be one over seven.
T: Then?
L5: And then continue two over seven three over seven and then four over seven.
T: What type of fraction is that?
L5: Improper fraction.

T: The type of fraction four sevenths?
L5: Four sevenths?
T: Yes. L5 could not answer the question. See 02:56-03:00v3; 03:00-03:36v3.
T: Okay let's continue. On the same number line, I want you to mark for me where fourteen sevenths is?
T: Where is fourteen sevenths?
L5: Here.
T: Where? At what number is fourteen sevenths?
L5: At number two.
T: What does it mean? Fourteen sevenths will simplify to what?
L5: To seven.
T: Seven? Fourteen divided by seven what is the answer?
L5: Ooo! The answer will be,
T: Will be?
L5: Will be seven.
T: Fourteen divided by?
L5: Two.
T: Do you see now?
L5: Yes
T: Let's move onto question (c). I want you to look at these circles. One circle represents a whole like one. And tell me what fraction is shown and write your answer an improper fraction. The answer L5 equated a mixed number 2 3/7 = 2 17/7. Though he explains the procedure of converting a mixed number to an improper fraction [which was the reverse of what he was asked by the interviewer/question], he claimed the equality above is correct. See 14:08-14:36v3. L5 did not write seventeen sevenths as a mixed number, see 14:40-15:40.
T: What fraction is at A?
L5: Will be four twoths [sic].
T: Write it here... What does four means?
L5: Four means a ... four means is a part on the ... of shaded parts in the number line to tell us the where is a point of four twoths.
T: Okay, how do you get two?
L5: They are the two whole. The teacher asked L5 to explain the meaning of a denominator four in the vernacular but he could not explain any better. See 18:00-18:52.
T: What type of fraction is seven eighths?
L6: A proper fraction.
T: What is a proper fraction?
L6: A fraction which consists of a numerator which is smaller than the denominator.
T: What else?
L6: And when you are drawing it on a number line, it is always before one. See 00:50-01:10.
T: Thank you very much. I want you to use your ruler and your pencil and you draw for me a number line, and put for me on that number line four sevenths. L6 was quick to draw and locate 4/7 on a number line with equal partitioning on a number line.
T: How do you know that it is there?
L6: Because from zero to one, because from zero to a whole, it is divided into seven equal parts and then I have to count where the four parts is and then I don't have to count zero because you

don't count the starting point, so I have to start from here and they are one two three four. So it is here. See 03:28-04:00.
T: I want you to indicate for me where is fourteen sevenths and what did you find out?
L6: Here, since fourteen sevenths is an improper fraction, the numerator is bigger than the denominator and then you have to change it into mixed fractions. When you change it into mixed fractions, it will give you two whole and here is two whole. See 04:16-04:35.
T: Thank you ... In question (c), the circle represents a whole and I want you to look at what shaded area ... How much of the shaded area represents a fraction?
L6: Seventeen sevenths.
T: Please write the same number as a mixed number.
T: Can you explain how a mixed number is then equal to parts shaded here and does the improper fraction relate to the diagram?
L6: For you to find the improper fraction and you are using this diagram, for the numerator, you count all the shaded parts in all the circles and they are seventeen. Here, they are seven and then plus this seven they are fourteen, then plus three they are seventeen. And then, the denominator you only count the parts in which one whole is divided into and they are seven. For a mixed number, you still have to use this one or use the diagrams. I want you to use a diagram. Since these two circles are completely shaded that means they are two whole and then count the remainder the one which is not whole shaded and then they are three parts which are shaded out of seven. See 05:36-06:33. T: Thank you. Please identify for me the fraction at point A and the fraction at B. L6 recorded A=4/6 and B=9/6.
T: Can you please tell me the difference between the fraction at point A and the fraction at B?
L6: The fraction at A is a proper fraction and it is before one and then the fraction at B is an improper fraction, it is after one. See 07:26-07:48.
T: Look at these two fractions 5/8 and 3/10, which one is larger? L3 drew number lines and use them to compare the two fractions appropriately.
L3: Five eighths is greater than three tenths because five eighths is closer to one, while three tenths is closer to zero. See 15:02-15:30.
T: I want you to order these fractions [3/5, 4/9, 2/7] in ascending order. L3 used fraction bars to order fractions correctly in ascending order. See 18:23-19:07.
T: Can you tell me which fraction is larger here?
L5: Three tenths.
T: Why?
L5: Because the numerators they are not the same. Because they are different in numbers.
T: Uhm!
L5: Five eighths is smaller than three tenths.
T: Because?
L5: Because there is a denominator ... there is three tenths there is bigger than five eighths, because the part of their cutting is ten pieces and then they shaded three parts and then become ... will be ... because ten has more parts than eight over five is smaller than three tenths, because five only eight pieces was equal and then shaded parts will be five. See 19:05-20:08v3.
T: Can you now use the greater than, equal and less than signs to show which fraction is greater?
L5 used the size of denominators to order fractions in ascending order [i.e.5/8<3/10 and 3/5<2/7<4/9]. See 21:00-22:04v3. L5 also added numerators and denominators together to obtain the sum [i.e. 2/5=2/3=4/8]. See 22:05-22:30v3.L5's experience was on defining fractions, see 22:40-22:45

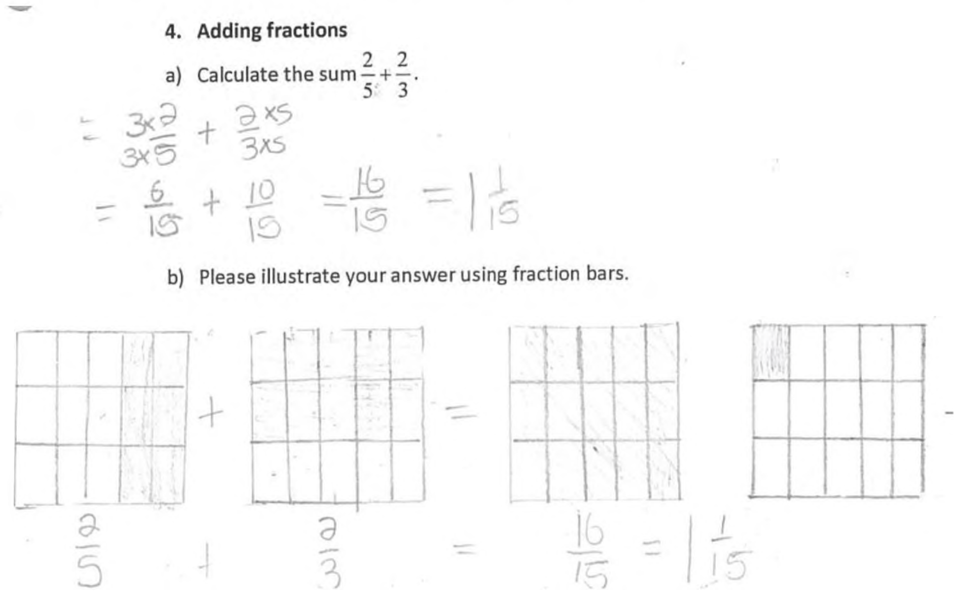
T: Let's look at comparing of fractions. Look at these two fractions five eighths and three tenths, can you please find out which one is bigger and insert the correct symbol between them. L6 used LCD to compare.
L6: Five eighths is bigger than three tenths. See 08:40-08:42.
T: How did you find out?
L6: Since the denominators are not the same, you have to convert them to get the LCDs so it will be easier for you ... You have to use the denominators, both the denominators, because there is no number which can multiply 8 and give ten and here there is no number which can divide into ten and gives you eight so you have to use the denominators. At the first fraction you use ten because it is a denominator here [at three tenths]. So you make denominator times ten and numerator times ten and then three tenths, you use eight so it will be vice-versa, because ten times eight is the same as eight times ten. So denominator times eight and numerator times eight. Five eighths will give you fifty over eighty and then three tenths will give you twenty-four over eighty. So now you compare because...
T: And now from there, what did you do?
L6: You just compare, because it is easier for you...
T: And now how did you compare this one and this one?
L6: Fifty eightieths is greater than twenty-four eightieths.
T: Because?
L6: Because here the numerator is ... no because twenty four is lesser than fifty.
T: What about eighty?
L6: They are just equal, so you have to look at the numerators. See 09:36-11:20. T: I want you to list those fractions from the smallest to the biggest in order of size. L6 used the number lines to determine the sizes of the given fractions after she asked for permission to use draw number lines to order fractions.
T: Just explain how you come to the answer?
L6: Three fifths is closer to a whole which is one, four ninths is closer to a half and two sevenths is closer to zero.
T: Okay.
L6: Since zero is the smallest and then comes a half and the whole so they will just follow each other in the sequence. The smallest will be two sevenths and then four ninths and then three fifths. See 14:32-15:36.
T: I want you to calculate the sum of 2/5+2/3.
L3 add numerators to numerators and denominator to denominator without finding the LCD [2/5+2/3=4/8]. L3 made use of one fraction bar only to show the sum but not the process of obtaining the sum of fractions using fraction bars. L3 did not use correct partitioning to show the process of finding the sum of fractions. Her understanding of fractions seems to be limited to the use of horizontal rectangular bar, especially on her first diagram of finding the sum.



Her experience of the program involved comparing fractions, which was the most thing that she found interesting and she claim that she learned how to use rules for comparing and ordering fractions such that for fractions with same denominators but different numerators, the smaller the numerator the smaller the fraction; and for fractions with the same numerators but different denominators, the smaller the denominator the bigger the fractions. See 29:00-30:07.

T: I want you to calculate the sum of two fifths and two thirds. L6 explained that one needs to use the LCD to add fractions. L6 explains that she has to change her answer of an improper fraction into a mixed number. The process of calculating and explaining were both correct. See 16:14-17:19.

T: Draw the fraction bars to show this addition, how to get the answer. L6 used fraction bars to show the process of obtaining the sum appropriately.



L6 explains well how to use the LCD to find the sum. She also explains well how she used fraction bars to show the process of adding fractions using the LCD. See 25:52-27:13. L6 said that the most salient thing in the program was comparing and ordering fractions especially the use of rules for comparing fractions. See 27:20-27:59.

APPENDIX H: TRANSCRIPTION OF RECALL INTERVIEWS

T: You said the answer for question 1.3 was C. What were you thinking?
L1: I thought these ones are divided into quarters, because that time I never knew the quarters. See 01:00-01:13.
T: What was the most difficult part to you?
L2: The number line. The number line, because that is why I drew this one. I did not understand, but now when you teach me, I understand that if you are drawing a mixed number on a number line, the number of the parts that are divided here [on a number line] they all have ... because you are drawing it as a mixed number you have to shade them and draw another whole which will give the fraction of that denominator, because you are given a mixed number. See 00:03-01:04.
T: What was difficult for you before?
L4: The number line, like to locate fractions on a number line like to find five tenths. I was only used to one over four, a half, then three quarters that was it, but now I know like on a number line how many lines should be on a number line to give you the denominator ... And for a fraction like two tenths, I need to count two times, one for the numerator and the other one for the denominator. See 00:03-01:06.
T: Can you show me something from your pre-test? You thought the answer is D, what was going on? How did you choose that answer?
L4: I thought two represents the denominator and the quarter of seven represents the numerator. See 02:00-02:25.
T: What was the most difficult thing to you from the beginning that you were struggling with and now you know?
L5: Defining fractions and drawing a number line.
T: Tell me, what did you learn that you did not know before?
L6: I learned how to draw fractions on a number line and how to compare.
T: Explain what were you thinking when you put three quarters here on a number line.
L6: I thought these numbers [referring to whole numbers] were the denominators and then they are one two three four so the denominator is four and then for you to get the numerator you just put the cross at three. See 01:34-01:56.
T: Tell me, what was the most difficult thing to you at the beginning?
L8: It was the number line especially to identify a fraction on a number line. See 00:18-00:27.
T: Here you say the answer is C [five quarters]. Is this true?
L8: I just forgot that a quarter is something out of four. See 00:48-01:26
T: What is that you did not know but now you know?
L9: How to show a fraction on a number line. Like here I was counting one over two one over three one over four one over five one over six... and thus I wrote here one over six over one. But, now I know that this is six ninths.
T: What else?
L9: I know how to find fractions using circles.... like here I thought when you are making a fraction, I thought you are going to count all the shaded parts and then the parts divided in a circle... Like here is seventeen over ... but now I know this one represents two whole and one over ninths.
T: What was the most difficult for you before the intervention?
L12: How to identify a fraction from a number line. See 02:40-02:50.
T: Tell me one thing that you learned that you did not know before?

L1: How to draw a number line.
T: What about a number line?
L1: I learned how to draw a number line.
T: Hmm?
L1: First when I wrote in the test I thought if as you are writing the pieces. See 00:35-00:45.
L4: And also finding or to locate a fraction on a number line, I used to get confused sometimes like sometimes I don’t know where to start with one over six one over five, so I used to get confused.
T: But, now?
L4: But then I only used to count like how many pieces from one and then I will get my denominator. It is like there are eight pieces in one and then I will start like one over eight, two over eight, three over eight, four over eight and five over eight like that.
T: But, then how did you get this one over five, you thought?
L4: Sir, I thought like, it will be like here I started like ... one over one, one over two, one over three, one over four and here one over five. See 02:27-04:10. *For question 1.8, L4 only used the last shaded parts from one to get one third, but the cancelled work shows that he only considered the shaded parts and total parts after one on a number line.*
L7: And also using the denominator of a fraction to work out the partitioning of a whole, like I divided one into ten parts instead of eight. See 00:20-02:00.
T: Here you got this answer, how did you get it? What were you thinking?
L8: I thought like there is a zero here it was confusing I did not know that from zero to one is the number that determine the denominator then I just found out that when you are counting from zero up to one you cannot count one two three four because here is one [pointing at one on the number line].
T: Hmm, you thought it is what?
L8: I thought is zero point and I did not know that it is out of eight so I counted like zero point one, zero point two, zero point three, then you can’t do that because one is already here. See 01:45-03:52.
T: And here how did get that may be the fraction three quarters is here?
L8: There I was confused sir, I just thought like this four maybe is... I use four and I counted from three and then this is one two three four.
L9: Six represents the shaded parts on a number line and then nine represents how many parts are divided in a whole. See 00:00-01:15.
T: How did you get thirteen over twenty?
L12: I counted the shaded parts which are thirteen and for twenty I counted all parts here [for 1.8 in the pre-test].
T: How did you decide to put three quarters there [for 1.7 in the pre-test]? What were you thinking? L12: I thought this is a whole.
T: And how did you get five eighths?
L12: I counted the number of parts. See 03:24-04:33.
T: For 1.7, you said your fraction three quarters is here. What were you thinking? How did you get this one?
L1: I got that one, because is where three is and then I thought on a number line if the numerator is for example is two, I can write it here [at two], because there is two. See also 01:20-01:50.
T: And now what do you know? What is the right thing that you think should be done?
L1: The right thing is three quarters is here [at the right position]. See 01:51-02:06.

T: What does that mean?

L1: It means three is the numerator and is less than one. See 02:07-02:42. T: Here what did you do?

L1: I forgot how I got this fraction. See 02:51-02:59.

T: Can you please tell me how did you get this answer, what were you doing so that you can get that answer?

L1: I was ... I thought this one together is twenty six, then I add twenty six plus twenty six ... I get fifty two and then I add this one again. See 03:00-03:50.

T: So you counted two times? Why did you count two times?

L1: I thought it is correct but now I know it is not, and I counted them all of them to get twenty six and then I counted all of them again and I added this one [thus she got fifty-three twenty-sixths]. See 03:52-04:13.

L5: And also reading mixed numbers using the area model and now I understand what a mixed number is.

T: What is a mixed number?

L5: It is a number that has a whole and the parts that are shaded as a numerator and the denominator ... See 01:03-02:10.

T: Show me please?

L6: How to draw fractions on a number line from fraction bars, you count the parts that the circles are divided into like here they are one, two, three, four, five, six, seven and when you are going to draw a number line.

T: Uhm?

L6: Since this is a proper fraction, it is only going to end until at one, because all proper fractions are before one and then between zero and one, you divide this part into seven equal parts ... you do not count from zero but from one two ... See 00:02-01:15.

T: Tell me, what was the most difficult thing for you?

L7: It was using drawings to show that an improper fraction is equal to a mixed number.

T: How did you get three tenths here [for 1.7 in the pre-test]?

L8: I start counting from here [pointing at 1], then I counted three for the numerator and ten parts of the denominator.

T: Here you got 1.3/2, how?

L9: I did not know how to find the fraction on the number line sir. But now the correct answer will be thirteen over ten which will give you one whole and three tenths. See part 03:15-04:16. See also part 04:20-05:10 on how L9 struggled to read a mixed fraction represented by the shaded area model of circles.

T: What is one thing that you find difficult before this program but now you have mastered?

L10: I used to count from zero on a number line.

L11: I also learned that improper fractions are always written after one.

T: How do you get that two sevenths is less three sevenths?

L1: Because ... first I thought that the numerator which is small is the bigger, then I decided three sevenths is bigger.

T: First when you said the numerator that is smaller is the one for the bigger fraction, give an example of how do you come to that? What were you thinking when you said if the numerator is smaller than the fraction is bigger? How did you see this number two sevenths?

L1: Two sevenths... I think [sic] seven are the parts that are the denominator, the parts that are divided in a whole, then two are the pieces that divided that are shaded. See 04:37-06:00.

T: And here what did you do? You said four fifths is less than four sixths?

L1: That is not correct.

T: Hmm? What did you think?

L1: I thought four sixths is bigger because the denominator is bigger. See 06:00-06:26.

T: And now, what is the correct answer?

L1: The correct answer is four fifths.

T: Okay, why four fifths now?

L1: Because the... the denominator which is smaller is for a bigger fraction.

T: And when is that true? When can you say that when the denominator is smaller, the fraction become bigger?

L1: When you draw it on the number line.

T: Okay? And when you look here, what do you see here? What is common in these two fractions? L1: When the numerators are the same but the denominators are different....

T: Then...

L1: Then you look for the lowest common factor.

T: And then?

L1: Then you are going to make four fifths times six over six. See 06:27-07:31 and 07:45-08:11.

T: And here you said two sevenths is greater than three sevenths, is that true?

L2: It is not true, sir.

T: What is wrong there?

L2: Because the denominators are same so you just consider the numerators. And this ... three sevenths is greater than two thirds, because if you draw it on a number line, the number of shaded parts that will be more will be the one for three sevenths and not two thirds. See 01:30-02:10.

T: And here [at 2.1.4 in the pre-test]?

L2: Four twelfths is greater than one third, because if you look at the number line four twelfths has more shaded number of parts and it is closer to a half of twelve which is six and one third is less than this and the numerator is closer to zero. See 03:10-04:10.

T: And this you supposed to order in ascending order? What do you say about your answer?

L2: It is not right.

T: Okay, because?

L2: Because five fifths is a whole, and this is just one piece from this twelve divided parts and this is just two [two thirds]. This one is closer to zero, it will be the smallest one followed by this one is closer to a whole, it will be the second and this is a whole, it will be the last one. See her first a See 04:30-05:19. T: Okay, and how did you get your answer before you knew this one method? L2: First, I thought that the fractions with smallest numerators are always the bigger ones that is why I got this one. See 05:55-06:29.

For question 2.1.4, L4 could not get the answer correct before the intervention, but he only realized that 4/12=1/3 after I asked him to simplify the fractions.

T: What else?

L5: Here is comparing fractions, you tell us to compare the numbers using the words and then we write in the words. Now I know what is less than and what is greater than. See 02:30-02:49.

T: What else?

L6: Ordering fractions... I did not know how to compare fractions when the denominators are not the same, but now I know that if the numerators are the same, you only consider the denominators. The smaller the denominator the bigger the fraction. Like here there is one quarter and a half and a half is bigger because its denominator is the smallest and therefore, it is the biggest fraction of all. And when the denominators are same, but the numerators are different, it is much easier, because the denominator determine the parts in which a whole is divided into, so it is just straight forward. See 02:28-03:25.

T: You said two sevenths is greater than three sevenths, what were you thinking? L8 claimed he knew the answer but just confused the signs of greater than and less than. See 05:31-06:19

T: What else?

L9: How to draw a fraction on a number line as well as how to use drawings for comparing fractions. See 01:30-03:12.

T: How do you know that one third is greater than one quarter?

L9: Because the shaded parts in all the…

T: In the fraction bars?

L9: One third is greater than one quarter. See 03:20-03:43

T: And here, you said 2/7 is greater than 3/7? Explain.

L9: I thought when the denominators are the same, the smaller the numerator the bigger the fraction. But now, when the denominators are the same, the bigger the numerator the bigger the fraction. And here when the denominators are different and numerators are the same, you have to look for the LCD. Here it will give me twenty four over thirty and twenty over thirty, so the bigger one is this one. See 05:30-06:30.

T: On ordering?

L9: I thought like when you are given one bread and then you have to divide it into eight parts, the other one in three parts and the last in five. I thought the ... the people who are going one who is going to get, ten people who are going to divide in three parts are the one who are going to get the biggest, and this one [eight tenths] is the smallest one. See 07:33-08:08.

T: What was difficult to you when we started?

L11: I did not know how to compare fractions using the fraction bars and using a number line.

T: What else did you learn?

L12: Comparing fractions using the fraction bars. See 04:43-04:56.

L12 use fraction bars to compare fractions in a pre-test. However, he said he did not learn this practice anywhere in school. This could explain why his fraction bars were not of equal dimensions. See 05:50-07:00

APPENDIX I: LESSON VIDEOS' TRANSCRIPTION

Lesson 1: Worksheet 1 (W1)

The teacher drew two fraction bars on the chalkboard. The first fraction bar was fully shaded and he explained to the class that it represent one unit like a loaf of bread. The second fraction bar had the same width as the first fraction bar but with a shorter length. The teacher asked the class to estimate the size of the second fraction bar in comparison to the first fraction bar.

L7: Seven over ten.
T: Or?
L2: Three over four.
T: These numbers are almost giving the same quantity. Let's look at these two fractions 'seven over ten' and 'three over four'. What does ten or four represent?
L8: Four represent a quarter. The teacher wrote one over four on the chalkboard and asked the class to confirm whether one over four was a quarter.
L8: Yes.
T: What does four tell you?
L5: It is the number of pieces.
T: The number of pieces where?
L6: The number of pieces in which a whole is divided.
T: Okay… But what does four tell us? It tell us the number of pieces that make up one whole unit. And how many pieces does this whole unit have?
L6: Four out of four.
T: How much?
L6: Four over four. The teacher was satisfied that learners now think of one unit [a fraction bar] as a fraction of unit subdivisions (i.e. 1=4/4). He drew a number line on the chalkboard and marked the position of zero, one and two. He asked the class to locate 7/10 on the number line by estimating. L3 went to the chalkboard and wrote seven on the number line between zero and one but closer to one.
T: What is that you wrote?
L3: Seven.
T: Look at the given fraction, is that seven? L3 erased 7 and wrote 7/10 correctly.
T: As you can see, the denominator is bigger than the numerator, what does that meant? If the fraction's denominator is bigger than the numerator then the fraction is less than?
L3: Ten.
T: Less than?
L3: Ten.
T: How can you write one as a fraction with a denominator of ten?
L12: You divide ten by ten.
T: Very good.

The teacher partitioned the distance between zero and one into ten equal parts and counted together with learners the parts by labelling each division after zero starting with 1/10 up to 10/10 which is 1. The teacher also divided the space between one and two into ten parts.

T: What is the next fraction after one?
L6: It is 1 1/10.
T: 1 1/10 is bigger than what?
L6: It is bigger than one whole.

T: Very good. Whenever the denominator and numerator are equal, then it is equal to one whole. It does not matter how many parts one whole unit is divided into, ne?
L6: Yes, sir. The teacher wanted to reinforce the relation between one, the fraction numerator and the fraction denominator, so he introduced a different fraction to test if learners understood this relationship.
T: If you have 7/8, is it less than one whole or more than one whole?
L6: Less than one whole.
T: Okay. Since 7/8 is not a full whole, is this fraction bigger than one?
L6: No.
T: Is it bigger than one?
L6: No, sir.
T: For you to have one full whole, how many pieces do you need?
L6: Eight.
T: Out of how many pieces?
L6: Eight.
T: Now in this activity [worksheet 1], I would like you to write fractions in words. For example, 3/4 as three quarters [he wrote both 3/4 and 'three quarters' on the chalkboard].
T: What is the number of shaded parts here [item one shows a fraction 2/3]?
L6: Two.
T: And what is the number of total parts in the whole unit?
L6: Three.
T: Ok. [2/3 was recorded on the chalkboard].
T: We are counting these parts because they are what?
L6: Three
T: No, no! We are counting the parts because they are?
L6: Equal.
T: Very good, and to write this fraction in words?
L1: Two over three.
T: Two over three or...?
L12: Two thirds
T: Good. In order to reinforce the importance of parts to be equal for them to represent a fraction, the teacher drew a fraction bar with four unequal parts and asked learners whether the four parts can be used to measure the fraction.
L9: No, because the parts are not equal.
T: Why?
L9: For them to be countable.
T: Yes. You can only count the parts when they are equal. A fraction measures the size of the area shaded which can only be determined once the whole unit is divided into equal parts.
Subsequently, the teacher encouraged learners to "avoid the use of the word 'over' when describing fractions because a fraction is one number and not two different numbers". Thereafter, learners were instructed to complete the rest of the items on the worksheet using the worked example.

Lesson 2: W2A

T: I hope you remember what you drew in the test? Did you understand how to draw those circles? L2: Yes Sir.
T: Ooo!

L2: The number line.
T: What? It is a problem, ne?
L2: Yes.
Learners struggled to use a compass to draw appropriate circles. Teacher asked them to use shorter periods on compasses and asked them to rotate the paper instead of compass to draw smooth, appropriate circles.
T: Help each other. L8 used one circle to show 5/4.
T: If you use one circle ne, you know you need five parts so one two three four, only four parts can fit on one circle. That’s why these four parts are equal to one circle. But you need five parts, so you need a second circle to show the fifth part. Teacher checked learners work as they were completing the worksheet. Teacher asked learners how they got their answers if they give inappropriate fractions.
L9: Sir!
T: Mhh!
L9: How did you get 11 over 4?
T: Say that again? How did I get one and a quarter?
L9: Yes.
T: It is one and a quarter. One full circle, which is one.
L9: Yes, sir.
T: So, now remember that each small piece is equal to what?
L9: A quarter.
T: So you have, a quarter, a quarter, a quarter, a quarter, which is a quarter plus one quarter plus one quarter plus one quarter, they will be how many quarters?
L4: Four quarters.
T: Plus another quarter?
L9: Five quarters.
T: This is five quarters, it is not five over four, but it is five quarters which is the same as one whole and a quarter. Is it not so? One whole unit represent one, ne?
L9: Okay.
T: Do you see that?
L9: Yes, sir.
T: Okay.
T: Are you done?
L7: Yes.
T: Did you verify everything?
L7: Yes Sir.
T: What did you write here?
L7: 11 out of 12.
T: Mhh? Are you sure?
L7: Yes, sir!
T: What type of fraction is this one?
L7: It is a proper fraction.
T: Is this proper?
L7: Ooo.
T: Mhh! L7 erased 11/12 and write 2 3/4=11/4 as an appropriate fraction.

T: Let’s quickly make a feedback! Can you just write down the fraction, in the last column? Hallow, fill this space.
L8: Yes, sir!
L4: Sir, what does this word meant?
T: It meant dividing. Partitioning comes from the word parts, so I just added the – ioning part.
L4: Okay, Sir.
L2: Haha, ooh, aye [no, no]! L2 erased her divisions of circles.
T: Mhh? What is wrong?
L2: Sir, I drew a lot of fractions.
T: A lot of what?
L2: A lot of fractions in this one.
T: Ooh!
L2: I divided them into fourteen parts.
T: And they are?
L2: And then they are just twelve.
T: Ooo!
L2: Aye.
L1 wrote 11/4 / 5/4.
T: You cannot use this sign (/). It is not right. It is not or, it is equal. Or, only write one of them. Is that clear?
L1: Yes, sir!
T: What is the fraction in 2?
L12: Is one.
T: Yes, the answer is one. Now how many circles should you draw?
L12: One.
T: And how many parts should be shown on that circle?
L12: Four.
T: Yee?
L12: Four.
T: Okay. Who can tell us why four?
L8: Because the denominator is being determined by how many parts are before one.
T: And how many parts are there before one.
L8: There are four parts.
T: Okay, one is divided in how many parts?
L11: Four. T: So any distance of one on the number line is divided into how many parts?
L11: Four.
T: Ne?
L11: Yes.
T: And that division of four parts in a whole unit should be maintained when dividing the circles.
L11: Okay.
T: Three? What fraction is that?
L6: Three over four?
T: Three over four?
L6: Three quarters.
T: Hee, three over four? Now you are not saying one thing, is like you have separate things, okay, three quarters, ne?

L6: Yes Sir.
T: And the quarter tells us how many times a circle should be divided into. You divide a circle into how many parts?
L1: Four parts.
T: Why four parts?
L1: Because the denominator is four.
T: And you got the denominator from where?
L1: From the shaded parts.
T: The shaded parts of what? Of a circle or of a number line?
L1: From the equal parts of the number line.
T: Okay. How many parts is one divided into a number line?
L1: Four.
T: Four, ne?
L1: Yes!
T: Okay, thank you very much.
T: What is the size of the piece in that whole circle?
L6: It is a quarter.
T: Yes, it is not a circle, but one quarter of a circle is missing.
T: Number four. What is the fraction for number four?
L10: It is thirty three twelfths.
T: Thirty three twelfths. Thank you very much.
T: How many circles should you have?
L6: Three.
T: And you should shade how many parts?
L6: Thirty three.
T: And in one circle, how many parts should be there?
L6: Twelve.
T: Okay, and in the last circle, how many parts you should shade?
L6: Nine.
T: Out of?
L6: Twelve.
T: Very good, you got it, ne?
L6: Yes!
T: Put something or draw something.
L4: Sir?
T: Yes, you made a mistake?
L4: Yes Sir. Here I supposed to draw nine.
T: It is not a problem.
L4: Okay Sir.
T: Yes. Okay and the last one? What is the fraction?
L4: It is eleven.
T: Eleven over?
T: Hey, you do not have a chance, let him talk. Read to us what you wrote there.
L4: Eleven eighths.
T: Eleven eighths?
T: Okay, is it eleven eighths?

L4: No Sir.
T: Okay, what is the fraction?
L4: Two whole and three become eleven.
T: Are you reading as a mixed number? Just give us one.
L4: Proper fraction?
L4: Three out of four.
T: Three out of four?
L4: Yes.
T: Three out of four is three quarters.
L4: Is not three quarters!
T: Hee? One learner: It is a mixed proper.
T: Hehe, okay, sorry. That fraction on the number line, what fraction is that? Is it a proper or improper fraction?
L8: It is a mixed number Sir.
T: It is a mixed number?
L8: Yes.
T: And a mixed number can be improper?
L8: Yes.
T: Okay, so it is not a proper fraction, ne?
L8: Yes.
T: Why is it not proper?
L6: Because the numerator is bigger than the denominator.
T: Mhh! The number of parts exceeds one. Those parts exceed one as a whole unit. What is the answer?
L4: Three over four.
T: What about two?
L4: Two wholes three over four.
T: Okay. Who can help us to read the answer correctly?
L2: Two and three quarter.
T: Yes. First you read the whole number and to connect them, you use the word ‘and’. What does the word ‘and’ implies?
L2: Yee [impressed].
T: In Mathematics, does it mean division … brackets or what?
L7: Plus
T: Plus, ne?
L7: Yes.
T: Yes, you are adding. So, we read two and three quarters. Is not so?
L7: Yes.
T: Okay, you should always use the word ‘and’. Very good. The last question.
L7: We did not do it.
T: Who did not do it? Learners showed by raising up their hands and they were many.
T: So we are going for the break.
T: What did you learn now from this activity?
L8: I learned how to draw the circles from the number line.
T: And what else?
L6: I learned that proper fractions cannot be changed into mixed numbers.

T: Okay. Thank you very much. What else?

L7: I learned that partitioning means dividing from the word parts.

T: Yes, partitioning means dividing an area into equal parts, ne?

L7: Yes.

T: What did you learn about the importance of the denominator when it comes to dividing the area of a circle? What does the denominator tell you?

L2: It tell you that, this number line or the circle must be divided into the number of parts that the denominator is showing.

T: Okay. Very good.

Lesson 2: W2B

T: Neat activity, you should quickly do those answers for the questions at the back. It is the same as the other one. *Teacher distributed the worksheets to every learner*.

T: Worksheets are to be collected and returned by Monday. *The teacher checked learners' work as they completed W2B.*

T: To make your life easier, you may use arrows to show the position of your fraction on the number line instead of shading like [L12].

T: You can see that drawing the number line is easier than drawing circles, ne?

Ls: Yes. T: Here I want you to define a mixed fraction, and here the improper fraction. Give me your definition of mixed number and how you will tell someone what an improper fraction or mixed number is, by also giving the properties of a mixed number.

Ls: Yes, Sir!

T: The last question is "when is it possible to give your answer as an improper fraction or as a mixed number?" That one is straight forward, I do not need to explain anything, and I hope it is very clear. Ne?

Ls: Yes.

Lesson 3: Feedback of W2B

T: Three over three is what? Three into three, goes how many times?

L2: Six.

T: Three into three?

L2: One.

T: What fraction is here? Write 2 as a fraction using these divisions?

L2: Two over two.

T: Two over two is one.

L2: Yes.

T: Noo, if you said two over two, then it will give one (2/2 =1). So what fraction is here?

L2: One.

T: No, I want you to use these divisions [on a number line]. It is not one. One is here. One is a whole number here. What fraction is here? You said, starts one third, two thirds, three thirds, four thirds, five thirds and here [at two]?

L2: A quarter may be.

T: Noo, a quarter shike ano [how]?

L2: Ooo!

T: Just continue counting.

L2: Five thirds, six thirds.

T: Write for me six thirds here. L2 Wrote 6/3.

T: Now simplify. 3 into 6 goes how many times?

L2: 2 times.
T: Yes! That is what I wanted you to see. That's why six thirds is 2. And to relate these.
T: Now I want you to write three, using the same divisions.
L2: You mean to continue counting up to three?
T: Yes. That is if you have 9 parts divided into 3, you will get?
L2: 3.
T: Yes and if you have 12 over 3, you will get?
L2: 4.
T: Yes. That is where the thing is coming from.
L2: Yes Sir.
T: This is not right. Each unit is divided into what? The teacher counted 12 divisions in a unit.
T: This supposed to be here [the arrow] at thirteen.
L4: Yes Sir.
T: You count twelve and then thirteen. So, this should be thirteen out of twelve, ne?
L4: Yes Sir.
T: Where did you get twenty four? All of them are shaded? Twelve plus twelve?
L4: Yes.
T: Then that's wrong. I am just correcting you, ne?
L4: Yes Sir.
T: This one you did it very nice. But only the equal spacing here should be made to be the same.
T: I am very happy with everything that you did.
L4: Yes Sir.
T: The only thing you need to improve on, is the use of rulers and equal partitioning. You also need to listen to the feedback, for this question.
T: I want to use these diagrams, to help you to see that your answer is going to be an improper fraction, ne?
L4: Yes sir.
T: What type of fraction is this?
L4: Three quarters.
T: Three quarters, what type of fraction is three quarters, proper or improper fraction?
L4: Proper.
T: Okay, proper. Look at the number of circles here and look at the number of circles here! All these answers are improper! Why? You see that, ne?
L4: Okay Sir.
T: Cool.
T: Next time, I do not want to see the shading, I want you to use the arrow to indicate the fraction on the number line. I am not saying shading is wrong, but this is the fast way to get the answer. L10: Yes, Sir.
T: You have done your best. Your number lines are very good.
T: [For L9], very good number lines.
T: You have done very well. Good division, and good use of pencil. Keep it up, ne?
L9: Yes.
T: You miss the centre. You see, all lines supposed to pass through one place. You see this one is passing here. They should pass through one place where the compass was standing.
T: You are very artistic, ne?
L12: Yes.

T: Therefore, very good boy.
T: Well done, but here, you only use a pencil. Next time, ne?
L3: Yes Sir.
T: Otherwise, your equal parts are here. I do not want us to shade, ne?
L3: Yes Sir. T: I want us to use arrows, ne?
L3: Yes Sir.
T: writing, just a pen to write your fractions ne.
T: I liked the way you looked at this one, ne?
L3: Yes sir.
T: This activity we are just looking at drawing, ne [talking to the class]. And it is going to help us to get the answers, ne?
T: So, she has just done her best to get the answer.
L3: Ha-ha [very happy].
T: Very neat work. Very nice one, very neat. This is what I want to see. Keep it up, ne? L6 smiled.
T: Good working. Keep it up, ne?
L8: Yes Sir.
T: Very good super star. You like this activity?
L8: Yes Sir.
L9: I do not know how to get from a mixed number to the improper fraction.
T: You do not know?
L8: Yes Sir. From the improper to the mixed number.
T: Okay, that's why we have these questions, to define mixed number and improper fraction. Let me see how you did it. You did very well.
T: Listen everyone, one of the skill I want us to learn in this activity is to use what you are given. Our method to get the answers, is, we use circles to find out what fraction is that one! When you look at the last activity for example when you look at this activity on the circles' side, all the fractions were improper fractions. Ne?
Ls: Yes.
T: Why they were improper? Because improper is when the fraction is greater than one. You can see that in all these, there is one full circle and additional circles. For example in question four, what is your mixed number?
Ls: 1 1/12.
T: This whole number in a mixed number, it tells you how many whole circles are shaded. How many are there?
Ls: One.
T: One. Now on the number line, it means all this distance [between zero and 1] is covered [shaded]. Ne?
Ls: Yes.
T: That means you got to shade from one onward. Ne?
Ls: Yes.
T: There is no way you can shade up to somewhere there [before 1 on a number line].
T: Because now you have one full whole [1 of 1 1/12], then what you do, you need to use two circles. Okay. For the second circle, you have one part out of how many parts?
L: Twelve parts.

T: Yes. So you have one full whole plus this part, together it become one whole and a twelfth! Ne?
Ls: Yes.
T: And now I want you to see a fraction and not parts for what you are counting, but as a size. That size which is one, and you get it by dividing the space of a whole. I meant a whole number. The space between two whole numbers. The space between two whole should be divided into equal parts and in this case they are?
Ls: Twelve.
T: They are twelve. Ne?
Ls: Yes, sir.
T: That's why the last question is there, look at question three. Question three asked! When is it possible to get your answer as an improper fraction or as a mixed number? It is when you are given fully shaded whole circle. At least you should be given one whole circle and another or more circles that are not complete.
T: What is your answer there?
L7: When you have a whole number and more.
T: Is it not so? When you look at the diagram, you should be given one complete circle or one complete whole and more.
T: I want you when you look at these diagrams, you should be able, to write your answers as a mixed number. The number of wholes, are shown as a whole and the number of shaded parts in the last circle are shown as improper fraction.
Ls: Proper.
T: Ne?
T: Neat, if I give you 3 1/5, how many circles do you need to draw this fraction?
Ls: Four.
T: Four?
LS: Yes
T: And the whole circle should be divided into?
Ls: Five parts.
T: And the last circle, how many parts should shaded?
Ls: One.
T: Like that. And if it was bread, you need how many bread?
Ls: Four breads.
T: How many breads is this?
Ls: Four.
T: No. How bread is this?
L6: Three breads.
Ls: Yes.
T: That answers the question of 'describe the process of getting a mixed number'. The process of getting a mixed number, you should check how many whole are shaded, to get a whole number for the mixed number. Ne? Check how many whole are shaded. And for the last circle [whole], check how many pieces are shaded in the last circle.
T: So, the number of whole circles shaded, to get a whole number for a mixed number. [Written on the board, and learners take note of this on their worksheet].
T: And the numerator of a proper fraction tell us how many shaded parts are in the last shaded circle [whole]. Is that clear?

Ls: Yes.
T: Okay, is the same now with the improper fraction. For the improper fraction, make sure that you count all the shaded parts in all the given?
Ls: Circles.
T: And then the denominator, the denominator should indicate how many times the whole should be divided into? Ne? Are you happy now?
Ls: Yes Sir.
L6: Sir?
T: Yes, ask me.
L6: When you are given 2 whole and they are all full shaded, do they fall under proper fraction, improper fraction or mixed number.
T: Thank you very much. The teacher drew 2 fully shaded circles.
T: When you are given 2 fully shaded circles, do you give your answer as proper, improper or as a mixed number?
L8: Proper.
T: Proper? How many whole are there?
L8: Two whole numbers.
L6: Sir, I do not ask for the answer, but I am asking if a mixed number is a proper fraction or improper fraction?
T: What is there [at 2 on a number line]?
L6: It is a whole.
T: Yes, and the answer is just a whole. And remember, why do we need fraction?
L2: Fractions are for counting things that are not whole.
T: Yes.
T: Another question.
L8: Sir, if you want to get an improper fraction from a mixed number without looking at the number line or circle, how will you get the answer?
T: Okay, who can help us?
L3: If the numerator is bigger than the denominator, check how many times the denominator goes into the numerator to get the whole and the remaining part will be the numerator and the denominator.
T: Okay. If you want to convert an improper fraction into a mixed number, without looking at the diagram, ne?
L8: Yes.
T: Give me any improper fraction?
L8: 7/2.
T: show 7/2 = (2+2+2+1)/2=3 ½.
Another example 17/5 = (5+5+5+2)/5 = 5/5+5/5+5/5+2/5 = 3 2/5.
T: Is it fine now?
L8: Yes, Sir.
L8: But what if the denominator was 2.
T: You like 8/2?
L8: Yes Sir. Take note 8/2 = (2+2+2+2+2)/2 = 4.
T: This means, if you have 8 halves, how many wholes? The answer is 4.
Lesson 3: W3
T handed out W3. T: I like the way you are thinking now. Can we read through?

Ls: No
T: Okay, I want you to identify the fraction on the number line. Learners worked on their own to complete W3!
T: People are very fast. For question 1 for example, just explain how you get the answer and you can use one of the fraction as an example.
T: You people are very smart. You guys are very good, this is what I want you to be able to do.
Lesson 3: Feedback of W3
T: What do you think was the purpose of this activity! Everything is on paper.
L6: Mhh!
L2: To identify fractions on a number line.
T: Yes. Now that you know how to draw number lines, this time they gave you the number line and now they want you to use your skill of counting to identify the fractions on the number lines.
T: What is the answer in a?
L5: Six tenths.
T: How did you get the answer?
L5: On a number line, it tell us in a, there is a mixed fraction.
T: Is at a?
L5: Yes.
T: Okay, now.
L5: We start at zero to ten. Then we say it become six out of ten because the first letter tells there are six out of ten.
T: Thank you, very much. What is the answer in b?
L1: One fifth.
T: One fifth, do you agree?
Ls: Yes.
T: Number three.
L12: Six eighths.
T: Do you agree?
L5: Yes.
T: Four?
L7: Two sixths.
T: Do you agree?
Ls: Yes.
T: Five?
L10: Six ninths!
T: Do you agree?
Ls: Yes.
T: Question b? Who want to speak English?
L9: All fractions are written in the proper fraction from.
T: Who can give us a complete answer?
L1: They are proper fractions become they have smaller numerators and bigger denominators.
T: Okay, yes?
L8: The fractions are smaller than one.
T: And therefore they are what?
Ls: They are proper fractions.
L6: Sir that is what I wanted to say!

T: Okay, say now what you wanted to say.
L6: They are all proper fractions. They are less than one. Their numerators are smaller than the denominators.
T: You are 120% correct. Can you just repeat what you said?
Lesson 4: W4 and W5
Ls: Sir, here I just…
T: What type of fraction is that?
L5: Proper fraction.
T: Okay, proper fraction. What are the properties of proper fractions?
T: We said the numerator is smaller than the denominator, what else? They are less than what?
L5: Less than one.
T: Okay, the denominator tell you what, about the division of the number line. You need to count the divisions starting from zero like one two three four five six seven eight and where there is eight you put one.
Ls: Okay, Sir.
T: And eight is the denominator.
L5: Okay.
T: Cool.
T: You do not need a pen to draw, the number line everything including the numbers and arrows.
L4: Yes Sir.
T: Are you doing fine?
L3: Yes Sir.
The teacher walked from one table of each learner to check how effectively they applied denominator to partition the number line and locate the fraction immediately appropriately. The teacher informed learners and L1 to complete the drawing of activity 5 at home.
T: Let's look at what we have achieved so far! Activity 4, was meant for you to integrate your understanding of a common fraction and whole number. I want us to understand that a fraction is a way of expressing things that are not in whole. And fractions, just like whole numbers, they also have a position on a number line?
Ls: Yes, on a fraction.
T: Okay, and the position of a fraction on a number line is determined by the numerator of that fraction. We also need to appreciate the difference between a proper fraction which does not have a whole, and a mixed number which has a whole number. Know also that improper fraction can be written as mixed fraction. Lastly, you should also work flexibly using a number line to identify a fraction, and use a denominator to partition the number line. I am happy that you are starting to think of fractions as a numbers.
T: Let's re-cap [using W4] how to draw a number line and indicate position of each given fraction. Can you convert the fraction in a as a mixed number?
Ls: No, Sir.
T: Give us a reason why?
L4: Because the numerator is smaller than the denominator.
T: And that fraction is less than one.
T: Yes, look on the number line, is it less than one?
Ls: Yes.
T: Therefore, we can conclude and say all proper fractions are?
Ls: Less than one.

T: And we then move to b. Can we write that fraction as a mixed number?
Ls: Yes Sir.
T: And what is that mixed number?
L8: Two whole and five out of ten.
T: Yes, two and five tenths, ne?
Ls: Yes.
T: What does that mixed tell you about how many whole numbers do you need to put on a number line?
Ls: 3.
T: And that fraction should be between what?
Ls: 2 and 3.
T: Very good.
T: Let's look at e. Can we write that fraction as a mixed number?
Ls: Yes.
T: What is the answer?
Ls: Four.
T: Just four, ne?
Ls: Yes.
T: Yes, the answer is just four, you can even look on the number line. The arrow pointed at which number?
Ls: Four.
T: That's why we used to say 12 divided by 3 is equal to what?
Ls: 4.
T: You now know where this things are coming from?
Ls: Yes.
T: And it is only true if the whole is divided into how many parts?
Ls: 3 equal parts.
T: Yes, if divided into 3 equal parts. The teacher then listen to L9's answer of the last question in worksheet 4.
T: Three quarter is between which whole numbers?
L9: Between ½ and 1.
T: Very good.
T: All the number in b are between which whole numbers? The learners continue in this way till all were done…
T: By looking at this fraction (1 1/3), you should be able to tell how the space between two whole numbers is divided into. The space between the whole numbers is divided into?
Ls: Three parts.
T: Three parts?
Ls: Yes.
T: Okay, and for 1 ¼, the space between whole numbers should be divided into?
Ls: Four parts

Lesson 5: W6A, W6B, W6C

The teacher check L12's W6A – L12 used lowest common denominator (LCD) to confirm his answer through he shaded. L8 used LCD method without shading fraction bars to compare fractions in W6A. L4 used LCD method to compare fractions inappropriately although he shaded fraction bars, he did not use them to compare. Then the teacher hand-out W6A, W6B and W6C to the class. L2, L4, L7 were teaching each other. L4 suggested the use of % to compare.

L2: Sir?

T: Yes?

L2: For example ... Ooo! These are eight people that are sharing one?

T: The number of people are down there. Parts are here! …

L2: Ooo!

T: Yes.

T: So what were you thinking?

L2: Sir, I was thinking that …. Because …

T: Okay, shade now, show five out of eight and shade 6 out of eight?

T: Which one is lager [after L2 shaded fraction bars while the teacher was looking]?

L2: This one [6/8].

T: Ooo! Then?

L2: Mhh!

T: You look at the amount shaded. You see!

L2: Mhh!

T: Yes, yes! *The teacher went to L4 to check his work. Later he left to another learner.*

T: I want you to look at these fractions. They have something in common. Then? Ooo! I thought is here. You have done very well.

L9: I do not understand here Sir?

T: What can you say about these fractions?

L9: They have a common denominator!

T: No, in your own words, what can you say about these fractions? Just look at 5/6 and 5/8, 5/6 > 5/8 because here you can see that the area for this one is greater than the area for this one.

T: Do not forget that you are comparing. If you are comparing, then what?

L12: Mhh!

T: I want you to explain why this fraction is greater than this one.

L8: ...

T: Then?

L8: ...

T: Mhh! Okay! And here?

L8: …to multiply this here…

T: Okay, I understand that one, but did you use these fraction bars to get your answers?

L8…

T: Can you shade these fraction bars for me? Can you shade for me one third here? Shade! That's what? You see? You see now! Check your answers! I put these things for you to check your answers. Which one is big?

L8: This one [1/3].

T: Good. Now, do the same with others.

T: [To the class], the sign should face the bigger number, ne?

Ls: Yes Sir!

T: Are you done?

L7: Yes Sir. The teacher explained the general question to L1.

T: Why is 2/5 > 1/5, what can you say about these fractions? What about five, and then and then, you are just fine, yes, write it down here. You have used, I want you to take two fractions from here and explain why this fraction is bigger than that one.
L1: Yes Sir.
T: This was peanut for you [L7]. The teacher read to himself what L7 wrote: The smaller the number…yes but ... May be the answer … since the numerator … are the same … check the denominators, the smaller the number, which number? Just change that to that.
L7: Yes, Sir…
T: You know these things now?
L7: Yes I do.
T: Ha-ha, and then?
L7: I converted to percentage.
T: Okay, but for example, let's look at this one … is more than...? You need to tell which one and then you just choose one of them. I just want you to explain. *The teacher went to L4's table*.
T: Use everything here to explain.
T: Are you done? How many minutes should I give you.
L3: 15
T: Okay, after 15 minutes, we should come and do the feedback.
T: Are you done with worksheet 6C? The teacher checked L3's worksheets.
T: Numbers are …. [Reading worksheets of L9]. L10's work seems not explicit and teacher spent time to make sense of her explanations.
T: Did you look at the fraction bars?
L10: No, Sir.
T: Why, we have been always looking at the shaded parts? Mhh? Do you see the mistake you made?
L10: Yes Sir.
T: Three tenths is just up to here, seven tenths is up to here. Which one is bigger?
L10: Seven tenths!
T: Okay, cool, then I want you to do the same for the rest.
T: How did you use the fraction bars to get this? Did you use the fraction bars to arrive at the answer?
L12: No!
T: Mhh? What did you shade there, how did you know if you did not? I want you to look at the fraction bars, and try to see some sense.
L12: Yes!
T: Where is this fraction here?
L12: Is this one!
T: And the other one?
L12: Is this one!
T: Okay which one is having more shaded area?
L12: Is this one.
T: And then, you did not even realized, you see. It is like that. These are here to help you to see that 5/6>5/8, because look at how the eighth is, this area is smaller comparing to the other one.
L12: Yes Sir.
T: That's what I wanted you to understand.
L12: Okay Sir.

T: How do you know that this one is bigger than that one? How do you know that 2/5 > 2/7?
L2: First, I look at the shaded area. The shaded part for 2/5 looks bigger than for 2/7.
T: Okay. The teacher read L2's explanation for the open ended question. The teacher read L1's explanation.
T: It is time to do business then we leave. The teacher copied all items in worksheets 6A-6C on the chalkboard.
T: When you look at the first activity over W6A, listen, I am going to help you how to answer this one and the purpose is to answer the question that says "what is special about comparing these fractions?
T: Those fractions have a common property and when you look at these ones [reading all fractions in W6A]. What is so special here? The numerators are what? Yes!
L8: The numerators are just the same then the other one is greater.
T: Okay, what did you look at?
L8: I was looking at the fraction bars.
T: Yes, what can you see on the fraction bars? Someone else.
L10: The numerators are the same. If some tell you, all these fractions look the same. Now, I want you to tell me how will you know which one is greater?
L10: You look at the shaded parts.
T: Yes, and shaded parts are?
L7: There in W6A, the numerators are the same, so you should look at the denominators. The smaller the denominator, the bigger the value of the fraction.
T: Hehe [laughing]. Thank you. Who want to say something else again? No learner raised up their hands. The teacher went to the chalkboard and talk to the class.
T: Same numerators, different denominators. Now which one is bigger, 1/4 or 1/3?
Ls: 1/3.
T: 3/4 and 3/9, which one is bigger?
Ls: 3/4 …
Ls: 2/5.
T: Four fifths or four sixths?
Ls: Four fifths.
T: Seven tenths and seven sevenths?
Ls: 7/7.
T: Now, when you look here, the denominators are different, so we choose the one which is smaller denominator which is 7/7.
Ls: Yes.
T: When you look at these fractions, they have the same numerators but the sign face the fraction with the smallest denominator.
Ls: Yes.
T: Here we have, we see the numerators are the same so, we take the fraction with the smallest denominator.
Ls: Yes.
T: Like here, 5 is smaller than nine?
Ls: Yes.
T: 3 is smaller than four?
Ls: Yes.

T: But the numerators for the first one are the same, for the third are the same and for the fourth one are the same.
Ls: Yes.
T: Now, what can you conclude? When you have the same numerators, different denominators, you can say that if the numerators are the same, the smaller the denominator the bigger the fraction [The teacher wrote it on the chalkboard and learners copied].
Ls: Yes.
T: I have used fraction bars to help you to compare, ne?
Ls: Yes.
T: But this is what I wanted you to look at when you look at these fractions, they are very special because in all cases, these fractions, the numerators are the same!
Ls: Yes!
T: And we found out that when the numerators are the same.
Ls: Yes.
T: Look at which denominator is smaller. It will save you energy and time. This is what I want you to learn now and write in the space here. And never forget. So when fractions has the same numerators, the one with a smaller denominator is the one which is bigger.
Ls: Yes.
T: And we can see this on the fraction bars, ne?
Ls: Yes.
T: Okay, let's go to b. Almost the same as a, the denominators are all the same. Which one is bigger here in a?
Ls: 2 over five.
T: Here?
Ls: five eighths.
T: Five eighths?
Ls: No, six eighths.
T: Here?
Ls: Seven... The class continued till the last item.
T: The sign should face the bigger fraction, ne?
Ls: Yes Sir.
T: Now, what is so special about these fractions? Who want to become the champion? You want to try, it is just almost the same.
L10: If the denominator are the same, the bigger the numerator, then the bigger, the fraction value. T: Yes, and thank you very much. [The teacher wrote L10's answer on the chalkboard as follows]: If the denominator are the same, then the bigger the numerator, the bigger the fraction / value.
T: Okay, we can say if the denominators are the same, the smaller the numerator, the smaller the fraction, ne?
Ls: Yes.
T: This one is for b not for a?
L4: Yes.
T: Why are writing it here?
L4: Sir, the first one, I write here.
T: Okay. The teacher read fractions to the class.
T: Which one is bigger here?

Ls: 1/2.
T: This one?
Ls: Equal.
T: This one?
Ls: 2/6 is greater than 2/3.
T: Okay. Ls: 4/5.
T: Is greater?
Ls: Yes.
T: Did you shade?
Ls: Yes.
T: The most important is that you should try to visualize these things, like how much is 3 sevenths, to be able to see which is bigger. Now, what can you say about these fractions? What is so?
L7: Check which fraction is closer to a half or a whole.
T: Mhh!
L7: If both fractions are past the half, you check which one is closer to a whole and if they are between zero and a half, you check which one is closer to a half.
T: This is 50% [referring to1/2 in a] and this is three quarters, ne?
Ls: Yes.
T: How can you compare these ones to a half or to a whole?
L12: Is less than a quarter.
T: Less than 1/4? Hee?
L12: Yes Sir.
T: You are saying 1/4 >3/4? Is that what you are saying?
L12: No Sir.
T: Okay, just correct the sentence / statement…
T: Is it true that 1/4 > 3/4?
Ls: No.
T: Okay, when you look at the fraction bars, 1/4 is less than a half. And 1/2 is equal to a half.
T: Now, when you look at the numerators and the denominators, there is no similarities, the numerators are different, the denominators are different. You need to compare which one is closer to zero, which one is closer to a half, which ones are closer to three quarters and which one is closer to one. Now when you look at this one [like] is closer to a half, is it not so?
Ls: Yes.
T: But a half is smaller than the fraction.
T: How do you know that 2/2=6/6.
Ls: We simplify.
T: Okay, you see now 1=1.
Ls: Yes.
T: How do you know which one is bigger?
Ls: You simplify.
T: Okay, let's simplify.
T: Now, which one is bigger?
Ls: 2 thirds.
T: Okay.
T: And this ones? 5/6 and 5/7, how do you know which one is bigger?

T: Which one is closer to a whole?
Ls: 6/7 is closer.
T: Yes! 6/7 > 5/6. So, here, when the numerators and denominators are different, use a half, or zero.
T: We want to finish. The teacher wrote on the chalkboard: When the numerators and denominators are different, then check how a fraction is closer to 1/4, 1/2, 3/4 or 1. That is the answer to open ended question in W6C.
Lesson 6: W7
T: Do we all understand that 5/4 > 6/5, ne?
Ls: Yes Sir!
T: Now I want a good answer. We have never used decimals in this class. Because decimals are for the calculators, they do not give a good understanding.
Ls: Sir we are done.
T: You are done?
T: What are you guys saying? The teacher talk to the group of L7.
T: Did you use the number line? Did you use zero or two? I want you to use the last method we used yesterday! When the number is closer to something?
Ls: Ooo! Sir? Sir?
T: Hee? Okay! You guys just wait for me, I will come to your group.
L3: Sir? I think…
T: Mhh, what do you think?
L3: We think 5/12 is a bit far from the whole and this one is a bit closer to a whole.
T: So?
L3: So, if it was inside.
T: No, no! Do not go that side, it is not that inside.
L3: So this one is a bit far from this one.
T: Then?
L3: Than this one. The teacher left to another group.
L3: Sir!
T: Okay!
T: What are you saying? Yes, very good, all of them are greater than one, then? They are equal?
L10: No Sir, they are not equal, the proper fractions… Then I said 5/4 is greater than 6/5 because it has one whole and one third.
T: Okay, I want you to check your reasoning that you wrote yesterday from W6C.
L10: Sir?
T: Yes, I got what you want to say, I want you to reason from a number line perspective, those numbers are closer to what, ne? Look like that!
Ls: Ooo!
L10: Sir, Is what I was telling you Sir.
T: You guys are fine. You are fine, do not worry.
L7: … while.
L3 smiled amazingly.
T: No, do not get angry.
L10: No, I am not angry Sir.
T: Then talk to me nicely. What are you saying? I understand you converted to mixed numbers there, you have zero, one and you have two, I want you to use those numbers to get your answer,

look at the last reasoning here which says check which one is closer what, what! I want you to say to which one of this, is this closer to? Okay!
Ls: Yes Sir.
T: Which one is closer to a whole?
L2: It is 5/4.
T: What about the other one. I want you to look here. What about the other, because when you are comparing you cannot just talk of one number, what about one other number. T: Let me go to this group, what are you saying?
L7's group: ……
T: Ijaa, what is wrong with you [tapping L7 on the shoulder for obtaining the correct answer], you wanted to bite me and now!
L3: Sir?
T: Hee? You, I said you are fine, ne?
L8: Yes, we are fine.
T: Yes, you have tried.
T: Guys let's look at the number line, ne?
T: This 6/5 is closer to the whole one.
T: Yes, no more a whole, this is one. From now when you use a whole, I hope you meant one.
T: Now, 5/4 is a bit far from one and that make it?
Ls: Bigger but 6/5 is a bit closer to one. Therefore 5/4 > 6/5.
T: Who can repeat what I said?
Ls: You look at number line, 6/5 is closer to one than 5/4 is far from one.
T: Therefore?
L5: Therefore 5/4 > 6/5.
T: Clap hands for him.
T: Who want to repeat what I said?
L8: Me Sir?
T: Okay.
T+Ls: 6/5 is closer to one while 5/4 is far from one, therefore 6/5 > 5/4.
T: Before you continue, where you are going to put your whole, this distance can diver but there wholes should always be equal. The one here and the two.
Ls: Okay Sir!
T: So, what do you do first, put your wholes zero, zero, decide where to put one first, otherwise you will get it wrong.
L7: Should we divide with a ruler or we should just use our hands?
T: You should try to use a ruler to make the parts equal.
L7: Yes Sir.
T: I want you to copy that one for number one and then you do the same for the rest. These things are not a lot. The reason for a is there, can you put the one for b?
Ls: Yes Sir.
T: Copy, I want you to have a good reason to have it on your paper.
L10: Sir, are we going to make a reason for each one?
T: Yes.
Ls: Uuh!
T: No, I am training you so that you can be okay. Because you were struggling with this one. They are not a lot. You only need a reason for b and c and for c in number 2.

L10: Are we going to give a reason for these ones?
T: No, only for b and c, I want you to give a short sentence how did you see that this one is bigger than the other fraction. For the first one, it is already given. Just give a reason for b and c and for this one.
T: I want you to use your number lines to reason.
Ls: Yes Sir.
T: Are you done?
L6: No! The teacher started to check and reaffirm learners' work while they were working. He kept guiding learners.
T: This one is closer to what?
L2: It is closer to a whole.
T: And since this is far from one… Learners were working out their answers but they needed much help and reaffirmation from the teacher to polish up their reasoning very often.
T: This number is closer to what? The teacher kept correcting learners reasoning.
T: Mhh, this is closer to what?
L7: To one.
T: Than what?
L7: Than….
T: This is closer to what?
L7: This is closer to one.
T: There is a lot of numbers there that you can use.
T: 5/4 is closer to one. This is closer to what?
L7: It is the same…..
T: Okay, it is fine.
L7: ...
T: Ooo, okay, that one, you are fine. Please copy the reasoning on the chalkboard.
T: What number is here?
L4: One.
T: And now, this one is closer to what?
L4: One.
T: Four fifths is closer to what?
L4: One.
T: This [4/5] is the one which is closer to one. This is closer to what? There are many numbers here, you can use zero, a half or use 3/4.
L4: Mhh!
T: Where is a half here?
T: That is one, where is a half of one?
L4: Is here!
T: Noo, is it here or there, where do you think a half is in b from zero to one?
L4: Is here!
T: Is this where one half is? Noo, show me where is the middle of that number line?
T: A half is here, you see.
L4: Yes Sir.
T: It is always there even if not shown.
T: Wow, you guys are doing very well. You are very smart. I wish you were already in grade 10, to give me As. People are doing their best.

T: You know what I advise you to do first, first you have to put one here, so that they can be straight and then you can start to make equal parts using a ruler, ne?
L8: Yes Sir. These numbers should be straight, ne? Then the only difference should be how many parts are here and how many parts are there.
L10: Sir, we need your help.
T: ___ is closer to a half while ___ is closer to one. Therefore 3/5 is greater. Ijaa, you guys speak well. The teacher read an answer for L11.
T: Because is closer to one while ………… Ijaa, you see you are talking now. Bring five. L2 also gave good reasons. The teacher then spends about 5 minutes showing L4 how to formulate good reasoning.
Lesson 7: W8A was not video recorded as the batteries of the camera were flat.
Lesson 8: W8B
T: Good Afternoon class.
Ls: Good Afternoon Sir?
T: You are asked to use the number lines to complete this activity. In the first one 4/10, 4/5, 4/8, you need to find the position of each fraction on the number line like what we did when we were comparing using a number line. Do you remember?
Ls: Yes Sir.
T: Then use your number lines to order the given fractions in ascending order. Remember, we have looked at how to order fractions using the fraction bars, ne?
Ls: Yes Sir.
T: Now, I want us to use the number lines to order. So, what we need to do is to draw three number lines, and all those number lines should be less than one. Because all fractions are less than?
Ls: One!
T: Okay, they are less than one. And for the first number line, you indicate where is 4/8, for the second number line, you indicate where is 4/10 and where is?
Ls: 4/5.
T: And then now you can see whichever is closer to zero, is the smallest one, the one in the middle is the one that come second and the one that is closer to one, is the one that come last! Ne?
Ls: Yes.
T: Can we do that?
Ls: Yes.
T: Thank you very much.
L2: No explanation today ne?
T: Explanations are just the same.
L2: Okay Sir.
T: Yes!
T: Let's start with the first one.
T: I want us to spend about seven minutes on each. If you draw your number lines and then indicate where each fraction is on that number line, and then you write a short sentence stating how did you know which one was smaller than the other on. I want us to be done by 4 O'clock. Let's try to be faster this time.
L2: Sir, should we also explain?
T: Yes, just write a short sentence. Use a ruler to draw your number line, ne?

Ls: Yes Sir. Learners worked on W8B individually, as seen in the video.
T: Aye, you are not doing anything, man! Please, finish number one first, then I will be happy! Ne?
L5: Okay Sir.
T: Okay, please!!! Put zero, one and one!
T: That is nice [to L7].
T: This is good [to L1].
T: Interesting [to L2].
T: This is very impressive of you guys, I am indeed grateful of that way you guys have changed so far [to L9 and L10].
T: This is very nice [to L3].
T: Do not forget the explanations, ne?
Ls: Yes Sir.
T: How many test, are you writing tomorrow?
Ls: Tests?
T: Yes.
Ls: We have for Accounting and Agriculture.
T: Tomorrow I won't be here, I am going to give my learners awards. What if we finish today, and next week Monday then we can only do the test. Let's have a class tonight. Then you can have sleepless nights up to nine o'clock, then we shall only remain with a test and interviews.
T: You are very intelligent [to L7]. L1 was called to order to only do one item per time.
T: People are speaking good English.
T: You did not answer the question. You did not list the fractions in ascending order. Please do that. Which one is small here? List them now!
L2: Mhh!
T: Yes, can you just do that now!
L2: Mhh!
T: I think we can start to talk now.
Ls: Yes, we can Sir.
T: Because we have to come back, ne?
Ls: Yes Sir.
T: Let's just look at one or two, since most of you are still at three. Is that fine?
Ls: Yes Sir.
T: I have seen that people have developed the skills of partitioning, is not so?
Ls: Yes Sir.
T: You people, have used the denominator to partition, ne?
Ls: Yes Sir.
T: In the first exercise, we have learned that the denominator represents how many parts a whole is divided into, ne?
Ls: Yes.
T: And then you guys are just doing that very well. Very good, keep it up. I am indeed impressed. Now, who can tell us how to arrange the first fractions 4/8, 4/10 and 4/5? L10: 4/10 < 4/8 < 4/5
T: So, which one is the biggest fraction here?
Ls: 4/5.
T: 4/5, ne?

Ls: Yes.
T: The sign always face the same direction when comparing and it should always face the bigger number. Why is that true?
L4: Sir, let me try.
T: Ooh, even me, I am trying.
Ls: Ha-ha.
L4: Sir I thought like the number of parts in a whole, you check that the more the number of parts in a whole, the more the fraction will be the smallest.
T: What are you saying?
L4: I am saying that the parts are many in a whole that means the fraction is small.
T: When is that true? Because it is not always true? When can we say that, we look at the denominators to say that?
L4: When their parts are the same.
T: Yes, when the number of shaded parts are the same. Who can give us his / her reasoning?
L6: Since all numerators are the same, I compared the fractions on the number lines according to zero, 1/4, 1/2 and 5/10 and 1. So, 4/10 is closer to five tenths which is equal to a half and 4/5 is closer to 1 and therefore 4/10 < 4/8 < 4/5.
T: Did you get what she said?
Ls: Yes.
T: Can you just repeat, and you should listen carefully now.
T: Her explanation I think is very rich! I think she said more than what one can ordinarily say. Hasho [right]?
Ls: Yes.
T: I thank you very much. Anybody else! I think the reasoning stays the same.
Ls: Sir, we did not finish Sir.
T: You did not finish.
Ls: Yes, Sir.
T: Who can give us the answer for b?
L5: 2/3 < 1/4 < 4/10 [L5 used denominators to order].
T: Who can give us a reason why is that correct?
L6: Me.
T: Can you give us a reason of how did you find the answer? Yes it is you.
L5: 2/3 is smaller because ….. Is far….
T: Just say what you have to say. Just say your reason, then we can help you.
L5: Because the denominators are not the same, therefore it was divided into equal parts.
T: Did you use 3 and 4 to arrange?
L5: To arrange.
T: Did you use 3 and 4 to arrange those fractions.
L5: Yes.
T: Okay, just explain.
L5: Because, I arrange 3, 4 and 10, the parts of two thirds will be equal to……
T: Yes.
L5…
T: Okay sit down. Who can comment on this answer?
Ls: It is wrong.
T: Okay. 4/10 is less than what?

L7: 1/4 < 4/10 < 2/3.
T: Okay, who can support the answer? Tell us why it is true.
L12: 1/4 is closer to zero, 4/10 is closer to a half and 2/3 is closer to 1. Therefore 1/4 < 4/10 < 2/3. L3: 1/4 is closer to 0 and 4/10 is closer to 1. Since the parts for 2/3 are bigger and then 1/4 < 4/10 and 2/3.
T: Okay.
L7: 1/4 is closer to 0, 4/10 is closer to 1/2 and 2/3 is closer to 1. Since 0 < 1/2 < 1, therefore 1/4 < 4/10 < 2/3.
T: Thank you very much. I think we can end here.
Ls: Yes. T: Thank you.
Lesson 9: W9
T: Good evening everybody?
Ls: Good Evening Sir.
T: I am very grateful for everything that God has done for us.
T: I am going to demonstrate what I want us to do. These two worksheets are just on additional worksheet. We have learned three methods of showing our understanding by drawing a circle or by using bars or using a number line, ne?
Ls: Yes Sir.
T: Now, when we add fractions, we should always check whether the denominators are the same, ne?
Ls: Yes Sir.
T: If they are not the same, what should we do?
Ls: We look for the lowest common denominator (LCD), or multiple.
T: Okay, now, without wasting much time, I just want us to understand where is this LCD coming from and I want us to use a picture to make sense why is it important to only add once we have the LCD, ne?
Ls: Yes.
T: Why should we make the denominators the same? Okay. From the first class we learned that we only count parts if they are of equal size, ne?
Ls: Yes.
T: Okay, let's have two examples that I am going to demonstrate with. Let's say
In a, 1/2 + 1/2 =2/2=1.
T: Now, we are adding fractions, ne? The first thing is to check if the denominators are the same ne? Are they the same?
Ls: Yes.
T: Yes. We can add the numerators and keep the denominators the same, ne?
Ls: Yes.
T: So, we can keep the denominator the same and then we can add one plus one, ne?
Ls: Yes.
T: And then we have?
Ls: 2/2.
T: Two over 2. We need to simplify which will give us?
Ls: One
T: One. Okay, let's look at b. We are given 1/4 +2/5, ne?
Ls: Mhh!

T: Now, for 1/4 + 2/5 the denominators are different, we cannot add. What should we do? Make them the same by looking for the LCD of 4 and 5, ne?
Ls: Yes.
T: Which is?
Ls: 20.
T: Twenty. Let's see. And how do you make four to become twenty?
Ls: By multiply 4 with five.
T: So, you have 1/4 x 5/5 + 2/5 x 4/4. Mind where you put your multiplication sign, make sure there is one division sign here and these are the same. 4 is coming from here, five is coming there. Then we apply BODMAS. T: BODMAS means check for brackets, divisions, multiplication, addition or subtraction.
T: Do we have brackets?
Ls: No.
T: Division?
Ls: No.
T: We have multiplication?
Ls: Yes.
T: Okay, before we work out addition, we should work out the multiplication sign, this one and that one. Then, 1x5? Ls: 5. T: 4x5? Ls: 20. T: 2x4? Ls: 8. T: 5x4=?
Ls: 20.
T: Are the denominators the same?
Ls: Yes.
T: Then, we can add and keep the denominators the same. 5/20 + 8/20 = (5+8)/20 = 13/20.
T: Why, does it really make sense for us to find the LCD?
Ls: Yes.
T: Let's show, now, how can we show that in a picture? Because, remember that you are comparing things, in reality, ne?
Ls: Yes.
T: Okay.
L7: Sir:
T: Yes.
L7: If you simplify 8/20 it will give you 2/5.
T: Is the denominator 1, numerator? It is a proper fraction, ne?
L7: No, but if you simplify it will become smaller.
T: What is your problem, now?
L7: No Sir, I am saying if you simplify 5/20 it will give you 1/4 but if you simplify 8/20 will give you two over five.
T: Hee [what now]?
L7: Ooo!
T: Thank you very much.
T: Let's say you have a bread and 2/5 of a bread. Together you have how much? Now, we need to use a diagram to represent that because that is reality. What should we do now? How can you people make sense of this? Do we really need twenty? Let's see. The teacher start showing learners using circles to show that 1/2 + 1/2 = 1.
T: So, you have a half, then you have another half. Now, a half plus another half, can give you one. But it can only works if these have same size. Ne?

Ls: Yes Sir.
T: Now you see that a half plus a half is really equal to one.
Ls: Yes.
T: Here the denominator did not change so…
T: Let's see now, whether it is important to find the LCD.
T: Let's say I have this bread. Then let's have a whole and divide it like this in four equal parts. And also divide the 2nd fraction bar into five equal parts. 1/4 + 2/5.
T: Are these parts equal to these?
Ls: Yes Sir.
T: Then you shade a quarter and 2 fifth here and have this. Now, are these equal?
Ls: No!
T: Okay, then we cannot count them together?
Ls: Yes.
T: Then we have a problem. What do you do is the following. To make these parts to become equal, you can say, let me say you can use five to divide these parts by drawing 5 five horizontal lines. Ne?
Ls: Yes, Sir.
T: Use this denominator to draw five horizontal lines here, ne?
Ls: Yes Sir.
T: Now, this one bar is having 5 shaded parts out of two equal parts. Use four to divide this area into equal parts as well. Ne?
Ls: Yes.
T: Now we see that ¼ + 2/5=5/20 + 8/20 =13/20.
Ls: Wow, Hee, awe!
L2: Sir, can you please repeat.
T: Okay, you can see in the first fraction bar 1/4 is in the here and then in the second one we have 2/5 shaded. Now, these parts are not equal. So, check the possibility to make them equal. By dividing these areas into equal parts. So, use the denominators to divide. So you can see that the area of one block here will be equal to the area of another block. Now you have 5/20 + 8/20 = 13/20. And when you look in the diagram of 5/20, you can still see 1/4 and in this one you can still have 2/5 for 8/20.
T: Now for 13/20, you only used the fraction like one of this. To get the answer, you need to know if the parts shaded are more than one whole, how many parts do we have here.
Ls: 13.
T: Are they more than a whole?
Ls: No.
T: Okay, then we only need one whole, ne?
Ls: Yes Sir.
T: You can shade 13 parts out of 20. The way of shading is not important.
L2: What if the fractions have the same numerators but different denominators, like 2/3 + 2/7?
T: Then you still look for the LCD, ne?
Ls: Yes.
T: 2/3 + 2/7 = 14/21 + 6/21 = 20/21. Then you shade 20/21 in the last fraction bar.
Ls: If I have 16/21 + 6/21 = 22/21 = 1 1/21. Then you will need one whole and more.
T: Thank you for asking. Learners start to work on item b, c and d of W9. Learners were given a choice to use circles or fraction bars.

T: First you do the calculation of the sum. After getting the sum, show your sum using fraction bars. So, the answer of the calculation should correspond to those you have on the fraction bars. That's why the LCD were introduced. The teacher explain why we used 20 as the LCM and not 40, 80 or 100. He wrote down multiples of 4 and 5 to show where the two is coming from.

T: So, the LCM is coming from the multiples of the given denominators.

T: Do you see that. It is easy to work with 20, because it is easy to work with 20 parts than having 40 or 100 parts. So, when you come for the interview, remember all these I have told you, ne?

Ls: Yes Sir.

L2: Sir, if I use 40, will I be marked wrong?

T: No, but you need to continue to simplifying which might be a lots of working for you.

T: It seems the second example is very interesting. L4 has 12 parts and 6 parts in separate fraction bars. L2 asked if they can use LCD of 18.

T: Yes, you can use 18.

L3: Ooo, okay.

T: But you have not calculated your answer?

L2: Yes, Sir. I am not yet done.

T: First I said calculate the sum before you draw, why are you going to the drawing?

L2: Sir, I calculated first then I erased.

T: What I am seeing here, I am not seeing the drawing, please finish this one first?

L2: Yes Sir.

T: Thank you.

T: You got the answer, ne?

L3: Yes Sir.

T: If the answer is not simplified, I need to ask you many questions.

T: Once you are done with b, you can move onto c. So, far when I was walking around, nobody is wrong. Just complete b the way you understand. I will come have one-on-one interview with everybody.

T: Do not copy from your neighbour. If you copy, you are a copy. But you are an original.

T: I am coming to you, and I am looking first at b. Is that b? I want you to delete this horizontal line.

L8: Yes, Sir.

T: And just delete that horizontal line. We are working together, you and him, ne?

L6: Yes Sir. L1 and L6 erased the line from the first bar. The teacher told L3 to delete the horizontal lines.

L3: Yes. The teacher walk around the class to tell learners who have the first fraction bar like L6, L8, and L3 to delete the horizontal line.

T: You see, this is 2 thirds and this is 1/6. You see ne?

L7 + L6: Yes.

T: However, it is not always that you need to cross multiply. You can see that in a third, you can have two of a sixth. So, what I can advise you is to create six parts in the fraction bar of 2/3. Just divide each third into 2 parts, then they become of the same size. You see that?

L8 + L6: Yes Sir.

T: Then you will have 4/6 + 1/6 and then even the area become the same. Can see 5/6? It is not always that you need to cross – multiply, it depends. You need to use your use your common sense.

T: Now, I want you to see, these are not corresponding.
L6: Ooo!
T: Now, you count, how many shaded parts?
L8: 4/6.
T: Ye [what]? Together?
L8: 5/6. T: Show me that they are five.
L8: Here there are four shaded parts and here there is one shaded part.
T: Together?
L8: Together they are five.
T: So, this denominator is what you draw in the fraction bar after the equal sign and then show me 5/6. I want to see that. Continue.
L8: Okay Sir.
T: Now, 15/18 is not simplified, is that the lowest common denominator?
L9: No Sir.
T: So, what is the LCD of 3 & 6!
L9: Six.
T: Okay, that means, you guys needs to get the LCD, is not so?
L9: Yes Sir…
T: So what should we do? Can I keep this paper of you?
L9: Yes.
T: Okay, thank you. The teacher collected L9's worksheet because it contains rich information on a learner' grappling with 2/3 + 1/6.
T: I want you here to draw 2/3 using horizontal lines.
L9: Yes Sir.
T: You got twelve? Mhh, interesting. Where did you get twelve? You multiply 6x2?
L10: Yes Sir.
T: Okay, then you multiply this one by 4, and then you get? What was your final answer?
L10: 10/12.
T: 10/12? So, how many parts are in a whole?
L10: 10.
T: No, the total number of equal parts?
L10: 12.
T: Okay, between 10 and 12, which one is bigger!
L10: 12 T: Now, is this fraction 10/12, a proper or improper fraction?
L10: Proper.
T: Can all these ten shaded parts fits in a whole?
L10: No Sir.
T: Ten parts?
L10: Yes Sir.
T: Now, why did you shade the second fraction bar after the equal sign? Now, you have here twelve but here you have 8. This paper of you I will keep it and I will give you another one, ne?
L10: Yes Sir.
T: Use vertical lines to partition the area of this horizontally partitioned fraction bars [to L10 and L9]. Then shade 2/3 for me here and then shade 1/6 in the other one.
T: You did not show your calculations, why? First, find the sum before you start using fraction bars.

L4: Okay Sir. T: But your parts are not the same, make them equal in both fraction bars.
L4: Yes Sir.
T: Is 18 the LCD?
L2: Is 12.
T: What about six?
L2: Mhh! Yes Sir.
T: And you? You have 15/18. It is not bad.
L1: Okay Sir.
T: Explain?
L7: Ooh, Sir. This one is.
T: Where is your calculation?
L7: I did it here.
T: Do not jump the steps. Explain now?
T: Yes.
L7: 6 becomes the LCD of 3 and 6.
T: Which is what?
L7: 6. T: Okay.
L7: Then I divided all into six parts. I have already shade 2/3 which becomes 4/6 and this 1/6 still remains the same, because the parts were already six. Now, I added four plus one then I get five out of six.
T: Okay. The only thing you did not do is to calculate the sum.
L7: How am I supposed to calculate the sum? The teacher referred L7 to the working on the chalkboard.
T: I will come back to you.
T: Are you happy now?
L8: Yes Sir.
T: Is the answer 5/6 again?
L6: Yes Sir.
T: Let me see, yes it is 5/6.
T: I first asked you to calculate.
L5: Yes Sir.
T: Where is the connection between this one and this one?
T: How many parts are here?
L8: They are six.
T: And here?
L8: They are 12.
T: Where is the LCD? L8: It is 6. T: Okay, how can you add these two fractions if you do not have a LCD? L8: Okay Sir. T: Okay, give me this paper and I will give you another one.
T: Wow, you have done very well.
T: Let's round up. I will show you another method to do b. T: 2/3 + 1/6, then what do you do?
Ls: Find LCD.
T: Which is? Ls: 6. T: So, this become 2/3x2/2+1/6=5/6.

APPENDIX J: RHODES UNIVERSITY PERMISSION TO CONDUCT RESEARCH

RHODES UNIVERSITY

Grahamstown • 6140 • South Africa

EDUCATION DEPARTMENT
Tel: +27 (0) 46 603 8383
Fax: +27 (0) 46 622 8028
PO Box 94, Grahamstown, 6140
E-mail: education@ru.ac.za

1 May 2015

To whom it may concern,

Dear Sir / Madam

PERMISSION TO CONDUCT RESEARCH

CANDIDATE: **Simon Albin**
STUDENT NUMBER: **13A6312**

This letter confirms that Simon Albin is a registered student of the Education Department at Rhodes University. He is currently registered for a Masters in Education.

In order to obtain this degree, Mr Albin will be required to conduct research in education. His research proposal was approved by the Education Higher Degrees Committee and complied with the ethical clearance requirements of the Faculty of Education at the meeting held on 23 April 2015.

The provisional title of the research was "Investigating the use of models to develop Grade 8 learners' conceptual understanding, adaptive reasoning and procedural fluency in fractions". This research project complies with the ethical clearance requirements of Rhodes University.

This letter serves to request permission for Mr Albin to conduct this research.

Yours Sincerely

Dr Bruce Brown
Head of Department
Education Department

APPENDIX K: REQUEST FOR PERMISSION TO THE DIRECTOR OF EDUCATION

RHODES UNIVERSITY
Where leaders learn

Enquiries: Mr. Simon Albin
Cell #: +26481 37 31570
Email: superalbin100percent@gmail.com

Education Department
Drosty Road,
Grahamstown 6139,
South Africa
18th May 2015

To: Mr. Xxxxxxxxxx
The Director of Education
Oshikoto Education Directorate
Private Bag 2028
Ondangwa

Dear Mr. Xxxxxxxxxx

REQUEST FOR PERMISSION TO CONDUCT RESEARCH STUDY

I am a fulltime Masters of Education student in the field of Mathematics Education at Rhodes University, Grahamstown, South Africa. To fulfil the requirements for the degree of Masters of Education, I am currently conducting a research project which aims to *'investigate the use of models to develop Grade 8 learners' conceptual understanding, and procedural fluency in fractions'*. This research study is solely a teaching intervention and I would very much like to implement and evaluate the impacts of intervention programme to Grade 8 learners in one of the secondary schools in your region. The research is planned to undertake from 25th May till 10th July 2015.

It is against this background that I am requesting your permission to conduct research study at one of the schools. Attached please find a copy of the confirmation letter from my supervisor, Dr. Bruce Brown who can be contacted at 046 603 8391 and b.brown@ru.ac.za.

Yours sincerely

Mr. Simon Albin
Med Student
Rhodes University

APPENDIX L: PERMISSION FROM THE DIRECTOR OF EDUCATION

REPUBLIC OF NAMIBIA

OSHIKOTO REGIONAL COUNCIL
DIRECTORATE OF EDUCATION, ARTS AND CULTURE

Tel (065) 281900
Fax (065) 240315
Enq: Mr Vilho Shipuata

Private Bag 2028
ONDANGWA
21 May 2015

Ref: 12/2/6/1

Mr Simon Albin
P O Box 2005
Rundu
Namibia

Dear Mr Albin

RE: PERMISSION TO CONDUCT RESEARCH

1. We acknowledge receipt of your letter dated 1st May 2015, seeking for approval from the office of the Director to conduct a research study in our Region.

2. The writing of this letter therefore serves to inform you that permission has been granted to you on the following conditions, that:

- You have to consult the school principals well in advance to make proper arrangements.
- The research should not interfere with the normal teaching and learning process at schools.
- And participation in the research should be on a voluntary basis.

3. With that in mind, it is my wish that your research study will yield satisfactory results, towards the completion of your qualification.

Sincerely yours

OSHIKOTO REGIONAL COUNCIL
Director
21 MAY 2015
Private Bag 2028
Ondangwa
DIRECTORATE OF EDUCATION

MR LAMEK T. KAFIDI
DIRECTOR OF EDUCATION
OSHIKOTO REGION

APPENDIX M: REQUEST FOR PERMISSION TO THE SCHOOL PRINCIPAL

RHODES UNIVERSITY
Where leaders learn

Enquiries: Mr. Simon Albin
Cell #: +26481 37 31570
Email: superalbin100percent@gmail.com

Education Department
Drosty Road,
Grahamstown 6139,
South Africa
18th May 2015

To: The Principal
Dear xxxxxxxxxx

REQUEST FOR PERMISSION TO CONDUCT RESEARCH STUDY

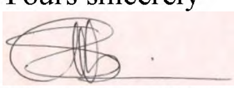
I am a fulltime Masters of Education student in the field of Mathematics Education at Rhodes University, Grahamstown, South Africa. To fulfil the requirements for the degree of Masters of Education, I am currently conducting a research project which aims to *'investigate the use of models to develop Grade 8 learners' conceptual understanding, and procedural fluency in fractions'*. This research study is solely a teaching intervention and I would very much like to implement and evaluate the impacts of intervention programme to a group of 12 Grade 8 learners in your school. In this regard I have, for the purpose of convenience, selected your school for my research study which I plan to undertake from 25th May till 10th July 2015.

This academic exercise is undertaken on the two folds of understanding first that the traditional teaching pedagogy of fractions contributes significantly to learners' difficulties in learning fractions and directly influences overall learners' achievement in mathematics in schools and beyond. Thus a new interactive teaching approach is chosen which is deemed effective to increase learners understanding and achievements in mathematics and fractions in particular. Lastly, research statistics reveals that learners with both conceptual and procedural knowledge outperform other learners. Hence, the objective of this research project.

It is my hope that the study will generate useful data, which will be valuable to different stakeholders in mathematics education. I further hope that participating in this study will be a learning opportunity for the participants. I undertake to uphold the autonomy of all participants and they will be free to withdraw from the research at any time without negative or undesirable consequences to themselves. In this regard, participants and their parents/guardians will be asked to complete a consent form.

It is against this background that I am requesting your permission to conduct research study at your school. Please feel free to contact me at any time should you have any queries or questions you would like answered. Attached please find a copy of the confirmation letter from my supervisor, Dr. Bruce Brown who can be contacted at 046 603 8391 and b.brown@ru.ac.za.

Yours sincerely



Mr. Simon Albin
Med Student
Rhodes University

APPENDIX N: PERMISSION FROM THE SCHOOL PRINCIPAL

SENIOR SECONDARY SCHOOL

PRIVATE BAG , TSUMEB TEL: (067) FAX: (067) E-MAIL: @iway.na

REFERENCE NO: 703

ENQUIRIES: **20 May 2015**

Dear Mr. Albin

RE: PERMISSION TO CONDUCT RESEARCH STUDY

Your request to conduct research for Mathematics Education hereby receives our favourable consideration. Thank you for choosing Secondary School for your research study.

Ms. , a Mathematics teacher, has been requested by the school to be your contact person and she will assist you in identifying learners for the research group as well as with any other logistics needed.

We look forward to working with you.

PRINCIPAL

MINISTRY OF EDUCATION
SENIOR SEC. SCHOOL
20 MAY 2015
Private Bag TSUMEB
Tel: 067 - / Fax: 067 -
OSHIKOTO REGION

APPENDIX O: REQUEST FOR PERMISSION FROM PARENTS

Enquiries: Mr. Simon Albin
Cell #: +26481 37 31570
Email: superalbin100percent@gmail.com

Education Department
Drosty Road,
Grahamstown 6139,
South Africa
3th June 2015

Dear parent/guardian of..

REQUEST OF PERMISSION TO PARTICIPATE IN A STUDY

This communiqué is solely an invitation for you to grant your genuine permission and your consent for your child of the above-mentioned name to participate in this research project. I am a fulltime Masters of Education student in the field of Mathematics Education at Rhodes University, Grahamstown, South Africa. To fulfil the requirements for the degree of Masters of Education, I am currently conducting a research project which aims to **'investigate the use of models to develop Grade 8 learners' conceptual understanding, adaptive reasoning and procedural fluency in fractions'**. This research program takes a form of a teaching intervention comprised of a series of lessons (to be taught during after morning school hours-sessions-in the afternoon) developed by the researcher to specifically help grade 8 learners intensify their fractions understanding as numbers, for comparing, ordering and adding fractions. I have confidence that his/her participation in this study shall give him/her best learning experiences that would enhance his/her understanding of fractions and exponentially tap at best his/her curiosity in studying mathematics successfully as a subject of his/her choice at school and beyond. In order to make best use of this learning opportunity, the program would give training lessons, rich group learning activities and both pre-test and post-test. The entire research is scheduled to commence from 09th June till 10th July 2015.

Therefore, I am requesting for your permission to allow your child to partake in this lifetime-rewarding opportunity of learning. Kindly complete and sign the attached declaration form for your consent. In addition, please take note that his/her participation in this study remains voluntarily, and so his/her identity shall remain anonymous and be treated confidentially always. Likewise, he/she reserve the right to withdraw from this study without any form of consequences. Please feel free to contact me at any time for any queries concerning this research.

Yours sincerely

....

Mr. Simon Albin
Med Student
Rhodes University

DECLARATION FORM

I, ... (Full names of a parent/guardian) hereby confirm that I understand the content of this document and the nature of the teaching intervention. Therefore, I grant permission for my child .... to become a participant of this research study.

.. **.......................**

Signature (Parent/guardian) **Date**

APPENDIX P: INVITATION LETTER OF RESEARCH PARTICIPANTS

Enquiries: Mr. Simon Albin
Cell #: +26481 15 01187
Email: superalbin100percent@gmail.com

Education Department
Drosty Road,
Grahamstown 6139,
South Africa
3th June 2015

Dear learner

INVITATION TO PARTICIPATE IN A STUDY

This communiqué is solely an invitation for you to participate in this research project. I am a fulltime Masters of Education student in the field of Mathematics Education at Rhodes University, Grahamstown, South Africa. To fulfil the requirements for the degree of Masters of Education, I am currently conducting a research project which aims to **'investigate the use of models to develop Grade 8 learners' conceptual understanding, adaptive reasoning and procedural fluency in fractions'**. This research program takes a form of a teaching intervention comprised of a series of lessons (to be taught during after morning school hours-sessions-in the afternoon) developed by the researcher to specifically help grade 8 learners intensify their fractions understanding as numbers, for comparing, ordering and adding fractions. I have confidence that your participation in this study shall give you best learning experiences to enhance your understanding of fractions and exponentially tap at best your curiosity in studying mathematics successfully as a subject of your choice at school and beyond. In order to make best use of this learning opportunity, the program would give training lessons, rich group learning activities and both pre-test and post-test. The entire research is scheduled to commence from 09th June till 10th July 2015.

Therefore, I am requesting for your permission to partake in this life-rewarding research study. Kindly complete and sign the attached declaration form for your consent. In addition, please take note that your participation in this study remains voluntarily, and so your identity shall remain anonymous and be treated confidentially always. Likewise, you also reserve the right to withdraw from this study without any form of consequences. Please feel free to contact me at any time for any queries concerning this research.

Yours sincerely

……………………..............

Mr. Simon Albin
Med Student
Rhodes University

DECLARATION FORM

I, ...……….…………………………………….. (Full names of a participating learner) hereby confirm that I understand the content of this document and the nature of the teaching intervention. Therefore, I grant permission to become a participant of this research study.

……………………..….......... **…………..........……**

Signature (Participant) **Date**