AN ANALYSIS OF THE LIBOR AND SWAP MARKET MODELS FOR PRICING INTEREST-RATE DERIVATIVES

by

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Abstract

This thesis focuses on the non-arbitrage (fair) pricing of interest rate derivatives, in particular caplets and swaptions using the LIBOR market model (LMM) developed by Brace, Gatarek, and Musiela (1997) and Swap market model (SMM) developed Jamshidan (1997), respectively. Today, in most financial markets, interest rate derivatives are priced using the renowned Black-Scholes formula developed by Black and Scholes (1973). We present new pricing models for caplets and swaptions, which can be implemented in the financial market other than the Black-Scholes model. We theoretically construct these "new market models" and then test their practical aspects. We show that the dynamics of the LMM imply a pricing formula for caplets that has the same structure as the Black-Scholes pricing formula for a caplet that is used by market practitioners. For the SMM we also theoretically construct an arbitrage-free interest rate model that implies a pricing formula for swaptions that has the same structure as the Black-Scholes pricing formula for swaptions. We empirically compare the pricing performance of the LMM against the Black-Scholes for pricing caplets using Monte Carlo methods.

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Chapter 1

Introduction

Financial derivatives securities are instruments whose value depends on the values of more basic underlying assets, like the prices of other traded assets such as interest-rates, commodity prices or stock indices. Interest-rate derivatives are securities whose underlying variables or assets are interest-rates. The commonly traded derivative securities in the market today are forward contracts, options and swaps. An option is a contract that gives the holder the right, but not the obligation to buy or sell the underlying asset by a certain date called the maturity and for a certain price called the strike price. A call option gives the holder the right to buy the underlying security, whilst a put option gives the holder the right to sell the underlying asset. The option is said to be exercised when the holder chooses to buy or sell the asset. If the option can only be exercised at maturity, then the option is called a European option. Otherwise, if the option can be exercised anytime prior the maturity date, then the option is called an American option.

Interest-rate Caps and Caplets

An interest-rate cap is a financial contract where the seller (issuer) of the contract promises to pay a certain amount of cash flow to the holder of the contract if the interest-rate exceeds a certain predetermined level, called the cap rate (Bingham and Kiesel, [1]). An interest-rate cap can be viewed as a portfolio of caplets. A caplet is a call option written on the floating interest rate such as the London Inter-Bank Offered Rate (LIBOR). A caplet guarantees that the interest charged on the floating rate loan at any given time will be the minimum of the prevailing floating rate, say the LIBOR rate and a preset cap rate. If the rate rises above the cap rate, the holder receives cash flow from the issuer which exactly compensates the additional interest expense incurred beyond the cap rate; if otherwise, then no cash flow results (Kwok, [18]).

Interest-rate Swaps and Swaptions

A swap is a financial agreement between two or more parties to exchange a sequence of cash flows over a period in the future (Kolb, [16]) . A well known type of swap is the "plain vanilla" interest-rate swap, in which cash flows equivalent to interest are exchanged between a fixed-rate payer (floating-rate receiver) and a floating-rate payer (fixed-rate receiver), based on a notional principal and over the same period of time until maturity. The reference floating rate is usually the LIBOR rate. The fixed rate that makes the present values of the two cash flows equal is called the swap rate (Kijima, [14]). Since the swap rate can be observed in the market as the LIBOR rate changes, it can be treated as an underlying asset; hence an option written on the swap rate is known as a swaption. A swaption gives the holder the right, but not the obligation to enter into a specified interest-rate swap.

1.1 Pricing caplets and swaptions.

The interest rate derivative market is the largest derivatives market, both in variety and complexity in the world today. As a result a lot of research has been dedicated to the pricing models of these derivatives, in a quest for more efficient and robust pricing strategies. This thesis follows guidelines on the work done by other researchers on the arbitrage-free pricing of interest-rate options, in particular caplets and swaptions, which are of "European" type. Under this framework we construct pricing models for caplets and swaptions using the so called "new market models". These are the LIBOR Market Model (LMM) developed by Brace, Gatarek, and Musiela ([4]) and the Swap Market Model developed by Jamshidan ([12]). From the discovery of the renowned Black-Scholes model by Black and Scholes ([2]) in 1973, which was further developed by Robert Merton, it has been considered the standard model for determining the fair price of an option. The Black-Scholes model is a mathematical model based on the notion that prices of stock or underlying assets of the option follow a stochastic process which is log-normally distributed. The dynamics of the Black-Scholes model can be constructed using partial differential equations which is mathematically demanding.

Other pricing models for valuing options such as the Heath-Jarrow-Morton (HJM) model of Heath, Jarrow and Morton ([11]) where presented. The stochastic drivers for the HJM model are the instantaneous forward rates. The HJM model was constructed using stochastic differential equations. This model did not thrive that much in the global finance markets, as traders continued pricing caplets and swaptions using the Black-Scholes model. The major shortfall of the HJM model was due to the fact that neither the instantaneous forward rates nor their volatilities were observable in the market. This led to the development of the "new market models", the LMM and SMM. The dynamics of the LMM and SMM are constructed using stochastic differential equations as in the HJM model. The major advantages of these models are that, firstly they are constructed using the volatility structures of the forward rates e.g. LIBOR and swap rates, which are observable or quoted in the market. Secondly the forward rates are log-normally distributed. This implies that for the LMM, the forward LIBOR rates follow log-normal processes such that the dynamics of the LMM will have similar structure to the Black-Scholes pricing formula for caplets. Similarly for the SMM, the forward swap rates follow log-normal processes such that the dynamics of the SMM are similar to the Black-Scholes pricing formula for swaptions in structure. As a consequence the LMM and SMM can easily be calibrated to match Black-Scholes option prices by directly inserting the quoted implied Black-Scholes volatilities into these models. LMM and SMM construction will be done in the continuous-time framework.

In chapter 2 we discuss the basic setup for the continuous-time securities model. We detail the concept of arbitrage-free pricing of securities using martingale probability measures. We discuss concepts of risk-neutral and forward-neutral martingale measures. Lastly we introduce the setup of the Black-Scholes model. In chapter 3 we construct the LMM using the stochastic tools discussed in chapter 2. We prove the existence of the LMM as set of stochastic differential equations for the forward LIBOR rate under forward-neutral martingale measure. We use the dynamics of the LMM to theoretically price caplets, and we show that the pricing formula is similar in structure to the Black-Scholes pricing formula for caplets. In chapter 4 we construct the SMM using similar techniques to those used in chapter 3. We prove the existence of the SMM as a set of stochastic differential equations under the forward swap martingale measure. We construct a pricing formula for swaptions within the SMM, and we show that it is structurally similar to that of the Black-Scholes pricing formula for swaptions. In chapter 5 we look at numerical pricing of caplets using the LMM. This will be done using Monte Carlo methods. We transform the LMM from its continuous-time framework to its discrete counterpart, using Euler's scheme, for simulation purposes. We show that Monte Carlo prices generated for caplets within the LMM are similar to the exact prices of caplets generated using the Black-Scholes formula for caplets.

Chapter 2

Continuous-time model for derivative securities.

In order for us to construct the LIBOR Market Model (LMM) and the Swap Market Model (SMM) we require some background theory on pricing derivative securities in the continuous-time framework . A derivative security is a security whose price has primary dependence on the stochastic processes of the price of the underlying asset (Kwok, [18]). This means that the value of the derivative changes over time in an uncertain manner. Firstly uncertainity of security prices in the continuous-time framework is modelled using Brownian motions. In the discrete-time framework, uncertainity in the market is modelled using random walks. It can be shown that the random walk will converge by law to a Brownian motion using the generalized central limit theorem (See [14]). We show that Brownian motion is not ideal for modelling security price movements. Instead, we show that the time t price of a security can be modelled by geometric Brownian motion, and that the ratio of the stock prices are log-normally distributed. This is an assumption made for the famous Black-Scholes pricing model. Secondly the concept of quadratic variation of Brownian motion is briefly discussed. This will lead to the introduction of stochastic calculus such as stochastic integrals and stochastic differential equations with respect to Brownian motion, which are used to develop the continuous-time securities model. We also look at some important stochastic calculus tools, such as Ito's formula and the division rule. Lastly we give an outline of the general continuous-time model for the securities market. Concepts of replicable contingent claims, self-financing, absence of arbitrage and risk neutral methods are discussed. We note that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure. The reader can consult any textbooks of finance such as Etheridge ([8]), Ross ([25]), Neftci ([23]), Shreve ([28]), Kijima ([14]) and Kwok ([18]) for discussions that follow on continuous-time model for derivative securities.

2.1 Terminology and Notation

In general uncertainty in the financial market is modelled by stochastic processes. A stochastic process is a family of random variables $\{X(t),\ t \in T\}$ or $\{X(t)\}$ in short, parameterized by time $t \in T$. Let $(\Omega, \mathcal{F}, P)$ be a probability space under which $X(t)$ is defined on. Suppose that the time epochs consist of a closed interval $[0, T]$ where $T < \infty$ and the current time is denoted by $t = 0$. Let $S_i(t)$ denote the time t price of security i for $i = 0, 1, 2, ..., n$, where the initial price $S_i(0)$ is known by all investors .The securities price process is denoted by $\{S(t);\ 0 \leq t \leq T\}$ or $\{S(t)\}$ in short, where $S(t) = (S_0(t), S_1(t), ..., S_n(t))^\top$, $0 \leq t \leq T$. Let $\theta_i(t)$ denote the number of security i possessed at time t for $0 \leq t \leq T$, and $\theta(t) = (\theta_0(t), \theta_1(t), ..., \theta_n(t))^\top$ is the portfolio at that time.

Assumptions of the securities market

1. There are $n + 1$ securities in the market where $S_0(t)$ is a risk-free security, e.g the money market account or the default-free discount bond and $S_i(t)$, $i = 1, 2, \ldots, n$, being risky securities, e.g stocks.
2. The securities pays no dividends.
3. The market is assumed to be frictionless i.e.
   - No transactional costs or taxes.
   - All securities are perfectly divisible.
   - Short sales of securities are allowed without restriction.
   - Borrowing and lending rates of the risk-free security are the same.
4. All investors in the market are price takers i.e. their trading actions do not affect the probability distribution of the prices available.

Let $\mathcal{F}_t$ denote the information available on the securities in the market at time t. This includes all the information available from time $t = 0$ up until current time t such that, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t \subset \mathcal{F}$. The sequence of information $\{\mathcal{F}_t;\ t = 0, 1, ..., T\}$ or $\{\mathcal{F}_t\}$ in short satisfying the sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t \subset \mathcal{F}$, is called a filtration. $\mathcal{F}_t$ must contain at least the smallest σ-field generated from $\{S(u);\ u \leq t\}$.

Definition 2.1: A random variable X is said to be $\mathcal{F}_t$-*measurable*, or *measurable* with respect to $\mathcal{F}_t$ if the event, $\{x_1 < X < x_2\}$ belongs to $\mathcal{F}_t$ for any $x_1 < x_2$.

Remark: In the securities market setting, the time t price $S_i(t)$ must be $\mathcal{F}_t$-measurable for any choice of t.

Definition 2.2: A stochastic process $\{X(t);\ t \in T\}$ is said to be *adapted* to the filtration $\{\mathcal{F}_t\}$ if each $X(t)$ is measurable with respect to $\mathcal{F}_t$.

Remark: In the securities market setting the price process $\{S_i(t)\}$ is adapted to the filtration$\{\mathcal{F}_t\}$.

2.2 Brownian Motion.

Now that the assumptions of the securities market have been defined formally, we define the phenomenon of Brownian motion. Brownian motion is the stochastic driver for the continuous time securities model. It models the price movements of securities $\{S(t)\}$. Brownian motion is obtained as a limit of the discrete time random walk model. The risky securities $\{S(t)\}$ are modelled using geometric Brownian motion which is described later. Firstly we formally define standard Brownian motion (s.b.m.), then we extend the definition to Brownian motion.

Definition 2.3: Standard Brownian Motion

Let $\{z(t);\ t \geq 0\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{F}, P)$. The process $\{z(t)\}$ is called a *standard Brownian motion* (s.b.m.) if

1. It has independent increments.
2. The increment $z(t+s) - z(t)$ for all non negative t and s , is normally distributed with mean 0 and variance s, independently of time t.
3. It has continuous sample paths and $z(0) = 0$.

The s.b.m. is a special case of the Brownian motion. A stochastic process $\{X(t); t \geq 0\}$ defined by $X(t) = X(0) + \mu t + \sigma z(t)$ is called a *Brownian motion* with drift μ, diffusion coefficient σ. We note that $X(t)$ follows a normal distribution with mean μt and variance $\sigma^2 t$.

2.3 Geometric Brownian Motion.

Definition 2.4: Geometric Brownian Motion (see [25])

Let the present time be $t = 0$ and let $S(t)$ denote the price of the security at time t. Then the collection of prices, $S(t),\ 0 \leq t \leq T$, follows a *geometric Brownian motion* with drift parameter μ and volatility parameter $\sigma > 0$, if for all non-negative values of t and s, the random variable

$$\frac{S(t+s)}{S(t)},$$

is independent of all prices up to time t and if in addition,

$$log\left(\frac{S(t+s)}{S(t)}\right),$$

is a normal random variable with mean μs and variance $\sigma^2 s$.

The above definition implies that the series of prices will follow a geometric Brownian motion if the ratio of the price at time t in the future to the present price is independent of the past history of prices. Thus this ratio of prices is said to be log-normally distributed with mean μs and variance $\sigma^2 s$. A consequence of assuming that a security's prices follow a geometric Brownian motion is that once μ and σ are determined , it is only the present value - and not the history of past prices - that affects probabilities of future prices. Furthermore, for a given initial price $S(0)$, the expected value of the price at time t depends on both of the geometric Brownian motion parameters, μ and σ, and is given by

$$E[S(t)] = S(0)e^{(\mu+\frac{1}{2}\sigma^2)t}.$$

Thus under geometric Brownian motion, the expected price grows at the rate $\mu + \frac{1}{2}\sigma^2$. Geometric Brownian motion is also appropriate for modelling security prices because , it allows for non-negative security prices. The Black-Scholes model assumes that the dynamics of the time t price of a security $S(t)$ follows geometric Brownian motion .

2.4 Martingales

An important assumption when implementing the LMM and SMM, is that we need to ensure fair (no-arbitrage) pricing of swaptions and caplets. This can be achieved using the concept of martingales. The term martingale has its origin in gambling . It refers to the gambling tactic of doubling the stake when losing in order to recoup oneself (Kwok, [18]). In the studies of stochastic processes , martingales are defined in relation to an adapted stochastic process[1]. Initially we define martingales in the discrete time setting to describe its basic structure and then we extend it to the continuous-time setting.

Definition 2.5: Martingale (Discrete)

An $\{\mathcal{F}_t\}$ − adapted stochastic process $\{X(t);\ t = 0, 1, ..., T\}$, defined on probability space $(\Omega, \mathcal{F}, P)$, is called a *martingale* if

$$E[X(t+1) \mid \mathcal{F}_t] = X(t), \quad t = 0, 1, ..., T-1, \tag{2.1}$$

where the conditional expectation $E[X(t+1) \mid \mathcal{F}_t]$ can be written in short as $E_t[X(t+1)]$.

Remark: For the time t price of a security $S(t)$, definition 2.5 implies that $E_t[S(t+1)] = S(t)$. This means given the information $\mathcal{F}_t$, up to time t the expected future value, $E[S(t+1)]$ is the same as the current value $S(t)$.

Definition 2.6: Martingale (Continuous)

A continuous-time $\{\mathcal{F}_t\}$ − adapted stochastic process $\{X(t),\ t \in [0,T]\}$ is called a *martingale* with respect to $\{\mathcal{F}_t\}$ if ,

1. $E[|X(t)|] < \infty$, for each $t \in T$.
2. $E_t[X(s)] = X(t)$, for each $t \in T$ and $t < s \leq T$

<u>Example</u>: The standard Brownian motion $z(t)$ is a martingale with respect to filtration $\mathcal{F}_t = \sigma(z(s),\ s \leq t)$. The conditional expectation is given by $E_t[z(t+s)] = E_t[z(t+s) - z(t) + z(t)] = E_t[z(t+s) - z(t)] + E_t[z(t)]$. But the increments $z(t+s) - z(t),\ s \geq 0$, are independent of the information $\mathcal{F}_t$, therefore $E_t[z(t+s)] = E_t[z(t)] = z(t)$, thus indeed the standard Brownian motion is a martingale.

[1] See definition 2.2.

2.5 Stochastic Calculus

In order to construct arbitrage-free "market models" i.e. LMM and SMM, it is necessary to develop stochastic calculus tools that enable us to perform mathematical operations on functions of stochastic processes modelled by Brownian motions. Stochastic calculus is important for modelling the price process of securities because of the quadratic variation of a Brownian motion. In this section we firstly define the concept of quadratic variation of a Brownian motion. Secondly we discuss the necessary tools for stochastic calculus namely stochastic integrals, Ito's formula and stochastic differential equations. We consider the general case for the above as we shall apply them in the later chapters.

2.5.1 Quadratic variation of a Brownian motion.

Definition 2.7: Quadratic variation

The *quadratic variation* of a stochastic process $\{X(t)\}$ is defined, if the following limit exists,

$$[X, X](T) = \lim_{N \to \infty} \sum_{i=1}^{N} |X(t_i^N) - X(t_{i-1}^N)|^2, \quad \text{a.s.} \tag{2.2}$$

where the limit is taken over all partitions such that for each N , $0 = t_0^N < t_1^N < ... < t_{N-1}^N < t_N^N = T$ and $\delta_N = \max_i \{t_i^N - t_{i-1}^N\} \longrightarrow 0$ as $N \longrightarrow \infty$.

Proposition 2.1 *Consider the standard Brownian motion* $\{z(t)\}$. *Then the quadratic variation of* $\{z(t)\}$ *is given by*

$$[z, z](t) = t, \quad t \geq 0. \tag{2.3}$$

Proof

Let $N = 2^n$ and define

$$W_n(t) = \sum_{i=1}^{2^n} \Delta_{ni}^2,$$

where $\Delta_{ni} = z(\frac{it}{2^n}) - z(\frac{(i-1)t}{2^n}),\ i = 1, ..., 2^n$.

Now in order to prove proposition 2.1 we will show that

$$E[|W_n(t) - t|^2] \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

The increments of the Brownian motion $\{z(t)\}$ are $i.i.d$ and normally distributed, and

$$\Delta_{ni} \sim N\Big(0, \frac{t}{2^n}\Big), \;\; i = 1, 2, ..., 2^n.$$

Now

$$E[|W_n(t) - t|^2] \;\; = \;\; E[(W_n(t))^2] - 2tE[W_n(t)] + t^2, \tag{2.4}$$

where

$$E[W_n(t)] \;\; = \;\; E\Bigg[\sum_{i=1}^{2^n} \Delta_{ni}^2\Bigg] = \sum_{i=1}^{2^n} E\Big[\Delta_{ni}^2\Big] = \sum_{i=1}^{2^n} Var\Big[\Delta_{ni}\Big] = \sum_{i=1}^{2^n} \frac{t}{2^n} = (2^n)\frac{t}{2^n} = t,$$

and

$$E[(W_n(t))^2] \;\; = \;\; E\Bigg[\sum_{i=1}^{2^n} \Delta_{ni}^2\Bigg]^2 = \sum_{i=1}^{2^n} E\Big[\Delta_{ni}^4\Big] + E\Bigg[\sum_{j\neq k} \Delta_{nk}^2 \Delta_{nj}^2\Bigg].$$

But

$$\sum_{i=1}^{2^n} E\Big[\Delta_{ni}^4\Big] = \sum_{i=1}^{2^n} 3(\frac{t}{2^n})^2 = 3(2^n)(\frac{t}{2^n})^2 = \frac{3t^2}{2^n},$$

where we have used the well known fact that if $X \sim N(\mu, \sigma^2)$, then its fourth order moment is given by $E(X - \mu)^4 = 3(\sigma^2)^2$.

Moreover

$$E\Bigg[\sum_{j\neq k} \Delta_{nk}^2 \Delta_{nj}^2\Bigg] = \sum_{j\neq k} E\Big[\Delta_{nj}^2\Big]^2,$$

by the independence of the increments, therefore

$$E\Bigg[\sum_{j\neq k} \Delta_{nk}^2 \Delta_{nj}^2\Bigg] = \sum_{j\neq k} E\Big[\Delta_{nj}^2\Big]^2 = \sum_{j\neq k} Var\Big[\Delta_{nj}\Big]^2 = \sum_{j\neq k} \Big(\frac{t}{2^n}\Big)^2 = \sum_{j\neq k} \frac{t^2}{2^{2n}} = \frac{t^2}{2^{2n}}((2^n)^2 - 2^n).$$

Now combining the results obtained, it follows that

$$E[(W_n(t))^2] = \frac{3t^2}{2^n} + \frac{t^2}{2^{2n}}((2^n)^2 - 2^n) = \frac{3t^2}{2^n} + t^2 - \frac{t^2}{2^n} = \frac{2t^2}{2^n} + t^2.$$

Substituting the above results into equation (2.4) we get

$$\begin{aligned} E[|W_n(t) - t|^2] &= E[(W_n(t))^2] - 2tE[W_n(t)] + t^2, \\ &= \frac{2t^2}{2^n} + t^2 - 2t(t) + t^2 \\ &= \frac{2t^2}{2^n}. \end{aligned}$$

Now $E[|W_n(t) - t|^2] = \frac{2t^2}{2^n} \longrightarrow 0$ as $n \longrightarrow \infty$, which implies that $W_n(t) \longrightarrow t$ as $n \longrightarrow \infty$, and this proves proposition 2.1 that the standard Brownian motion $\{z(t)\}$satisfies the quadratic variation $[z, z](t) = t,\ t \geq 0$. ■

2.5.2 Stochastic Integrals

The Ito integral is a suitable mathematical tool for integrating a function that has non-zero quadratic variation such as the Brownian motion. It is one way of defining sums of uncountable and unpredictable random increments over time (Neftci, [23]). This section intuitively defines the Ito integral and we outline its properties. Before proceeding it is important to note two reasons leading to the practical relevance of the Ito integral. Firstly, security price dynamics with respect to Brownian motion are constructed by stochastic differential equations, which can be defined only in terms of the Ito integral. Secondly stochastic differential equations are defined on infinitesimal intervals and their use in finite intervals requires approximations. Such approximations will be defined by the use of the Ito integral.

Definition 2.8: Predictable Process

An adapted process[2] $\{\psi(t)\}$ is predictable, if it is left-continuous in time t, a limit of left-continuous processes, or a measurable function of a left-continuous process.

Remark: Any adapted and continuous process is predictable.

Now consider the closed time interval $[0, T]$. Partitioning it into n sub-intervals such that, $0 = t_0 < t_1 < ... < t_n = T$, define

[2] See definition 2.2.

$$I_n(t) = \sum_{i=0}^{n-1} \psi(t_i)\{z(t_{i+1} \wedge t) - z(t_i \wedge t)\}, \quad t \leq T, \tag{2.5}$$

where $t_i \wedge t = min(t_i, t)$ $i = 0, 1, ..., n$. Such an expression is meaningful in financial engineering, because it is directly related to martingales, one key notion in the theory of no-arbitrage pricing. The proposition below defines the Ito integral.

Proposition 2.2[3] *Let $\{\psi(t)\}$ be a predictable stochastic process, which satisfies the "non-explosive" condition, that is*

$$E[\int_0^T \psi^2(t)dt] < \infty.$$

Then $I_n(t)$ converges in mean square to the uniquely defined stochastic process $I(t)$, that is for $t \leq T$,

$$E\Big[I_n(t) - I(t)\Big]^2 \longrightarrow 0 \quad as\ \max_i(t_{i+1} - t_i) \longrightarrow 0.$$

The process $\{I(t)\ t \geq 0\}$ is called the Ito integral and the following intuitively appealing notation is used

$$I(t) = \int_0^t \psi(s)\, dz(s), \quad t \geq 0.$$

Properties of the Ito integral[4]

1. $I(t)$ is a martingale.
2. $I(t)$ has continuous sample paths.
3. $E[\int_0^t \psi(u)\, dz(u)] = 0$.
4. Ito isometry, i.e. $E[I^2(t)] = E[\{\int_0^t \psi(u)\, dz(u)\}^2] = \int_0^t E[\psi^2(u)\, du]$.
5. The Ito integral is normally distributed the mean 0 and variance $\int_0^t \psi^2(u)\, du$, i.e. $I(t) = \int_0^t \psi(u)\ dz(u) \sim N\Big(0, \int_0^t \psi^2(u)\, du\Big)$.

[3] See [14],[23],[18]and [28].

[4] Refer to Shreve ([28]) for detailed proofs of the properties listed below.

2.5.3 Stochastic Differential equations

In the literature of financial engineering, it is common to model a continuous-time price processes in terms of stochastic differential equations (SDE). In particular for the LMM and SMM, default-free discount bond prices, forward LIBOR rates and forward swap rates will be constructed using SDE's.

Consider the Brownian motion $\{z(t)\}$ with filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$. Let $\mu(X(t),t)$ and $\sigma(X(t),t)$ be adapted to $\mathcal{F}_t$ with $\int_0^T |\mu(X,t)|\ dt < \infty$ and $\int_0^T \sigma^2(X,t)\ dt < \infty$ (almost surely) for all T , then the process defined by

$$X(t) = X(0) + \int_0^t \mu(X(u),u)\,du + \int_0^t \sigma(X(u),u)\,dz(u), \tag{2.6}$$

is called an *Ito process.* The integral above can be written in its differential form,

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dz(t), \quad 0 \leq t \leq T, \tag{2.7}$$

or in short

$$dX = \mu(X,t)dt + \sigma(X,t)dz, \quad 0 \leq t \leq T, \tag{2.8}$$

where $\mu(X,t)$ is the drift function and $\sigma(X,t)$ is the diffusion coefficient and $X(t) = X$.

Remark: In financial engineering, to ensure arbitrage-free pricing, it is of fundamental importance that the price process $S(t)$ is a martingale. The next propositions (See [14]) provide important results on the properties of the SDE of a process which is a martingale.

Proposition 2.3 *Suppose that the process $\{X(t)\}$ is a solution to the SDE*

$$dX = \mu(X,t)dt + \sigma(X,t)dz, \quad 0 \leq t \leq T,$$

where $\sigma(x,t)$ is continuous and satisfies $E[\int_0^T \sigma^2(X,t)\ dt] < \infty$. Then the process $\{X(t)\}$ is a martingale if and only if the drift is zero, i.e. $\mu(x,t) = 0$.

Proposition 2.4 *Suppose that the process $\{X(t)\}$ is a solution to the SDE*

$$\frac{dX}{X} = \sigma(X,t)dz, \quad 0 \leq t \leq T,$$

where $\sigma(x,t)$ *is continuous and satisfies* $E[exp\{\frac{1}{2}\int_0^T \sigma^2(X,t)\ dt\}] < \infty$ [5]. *Then the process* $\{X(t)\}$ *is a martingale and satisfies the equation*

$$X(t) = X(0)exp\left\{\int_0^t \sigma(X,s)dz(s) - \frac{1}{2}\int_0^t \sigma^2(X,s)ds\right\},$$

whence it is called an exponential martingale.

Now the theory of stochastic integrals and stochastic differential equations can be extended to security price processes, $\{S(t)\}$. This is because in the continuous-time framework security price processes are modelled using Brownian motion which is not differentiable, almost surely. In financial theory it is common that we consider the rate of return of a security $\{S(t)\}$, rather than its price process directly. The rate of return of a security, is just the ratio of profit or loss on a security investment relative to the amount of money invested. Let $R(t)$ denote the time t instantaneous rate of return of a security $S(t)$. Assuming that the security does not pay dividends, then the rate of return of the security is given by

$$R(t)dt = \frac{dS(t)}{S(t)}, \quad 0 \leq t \leq T.$$

Assuming that this rate of return $\{R(t)\}$ is expressed in terms of a deterministic trend and unanticipated "noise"[6] then the rate of return of the security $S(t)$ is given by

$$R(t)dt = \frac{dS(t)}{S(t)} = \mu(S,t)dt + \sigma(S,t)dz, \quad 0 \leq t \leq T,$$

that is

$$dS(t) = \mu(S,t)S(t)dt + \sigma(S,t)S(t)dz,$$

or simply

$$dS = \mu(S,t)Sdt + \sigma(S,t)Sdz, \tag{2.9}$$

where $\{z(t)\}$ is a standard Brownian motion. The trend or drift function $\mu(S,t)$ is called the mean rate of return and $\sigma(S,t)$ is called the volatility of the security $S(t)$. The "noise" is given by $\sigma(S,t)dz$. In the sub-section that follows we look at an important mathematical tool for financial modelling, known as Ito's formula.

[5] This condition is called the Novikov condition.

[6] We assume that the "noise" around the security prices is modelled using the standard Brownian motion.

2.5.4 Ito's formula.

Ito's formula which is also known as Ito's lemma can be viewed as an extension to the chain rule in ordinary calculus. It plays an important role in stochastic calculus for financial engineering. For example consider the case where we have a stochastic process $\{Y(t)\}$ obtained from the process $\{X(t)\}$, which is a solution to the SDE (2.8), via a smooth function $f(x,t)$. The process $\{Y(t)\}$ is given by $Y(t) = f(X(t),t), \quad 0 \leq t \leq T$. Ito (1994) developed the following theorem for determining the SDE for $\{Y(t)\}$ and it is stated below, as cited by Kijima ([14]).

Theorem 2.1. Ito's formula

Let $\{X(t)\}$ be a stochastic process that satisfies the SDE

$$dX = \mu(X,t)dt + \sigma(X,t)dz, \quad 0 \leq t \leq T.$$

Then for a smooth function $f(x,t)$, $Y(t) = f(X(t),t)$ satisfies the SDE

$$dY = \mu_Y(t)dt + \sigma_Y(t)dz, \quad 0 \leq t \leq T,$$

where $\mu_Y(t) = f_t(X,t) + f_x(X,t)\mu(X,t) + \frac{1}{2}f_{xx}(X,t)\sigma^2(X,t)$ and $\sigma_Y(t) = f_x(X,t)\sigma(X,t)$.

Remark: The functions $f_t(x,t), \quad f_x(x,t)$ and $f_{xx}(x,t)$ are partial derivatives of $f(x,t)$ with respect to t and x respectively.

Proposition 2.5 (Division rule) *Consider the two SDE's*

$$\frac{dX}{X} = \mu_X(t)dt + \sigma_X(t)dz, \quad t \geq 0,$$

and

$$\frac{dY}{Y} = \mu_Y(t)dt + \sigma_Y(t)dz, \quad t \geq 0.$$

Let $D(t) = \frac{X(t)}{Y(t)}$ where $Y(t) > 0$, then the process $\{D(t)\}$, if it exists, follows the SDE

$$\frac{dD}{D} = \mu_D(t)dt + \sigma_D(t)dz, \quad t \geq 0,$$

where $\mu_D(t) = \mu_X(t) - \mu_Y(t) - \sigma_Y(t)(\sigma_X(t) - \sigma_Y(t))$ and $\sigma_D(t) = \sigma_X(t) - \sigma_Y(t)$.

2.6 No-arbitrage Pricing and Risk Neutral evaluation method.

In order to fairly price contingent claims such as caplets and swaptions it is of fundamental importance that there are no-arbitrage opportunities in the market i.e. a risk-free way of making profit. In this section we introduce the fundamentals of no-arbitrage pricing of contingent claims. A contingent claim is a random variable X representing a payoff at some future time T . In particular for this thesis, the future payoff functions for swaptions and caplets can be viewed as contingent claims. Firstly we define the concepts of self-financing and replicating portfolio's. An investment is said to be self financing if no extra funds are added or withdrawn from the initial investment. The cost of acquiring more units of one security in the portfolio is completely financed by the sale of some units of other securities within the same portfolio (Kwok, [18]). Secondly we show that no-arbitrage pricing implies the existence of a probability measure that makes the time t denominated price of a security a martingale. Such a probability measure is known as the risk-neutral probability measure. A risk neutral investor is one who values an investment solely through the expected value of the securities return (Ross, [25]). For the definitions and explanations that follow, we assume that the securities market $S(t)$ pays no dividends[7].

Now consider the portfolio process $\{\theta(t)\}$ defined in section 2.1. Then the value process $\{V(t)\}$ is defined by

$$V(t) = \sum_{i=0}^{n} \theta_i(t) S_i(t), \quad 0 \leq t \leq T, \tag{2.10}$$

which can be expressed in the differential form

$$dV(t) = \sum_{i=0}^{n} \theta_i(t) dS_i(t), \quad 0 \leq t \leq T, \tag{2.11}$$

or the integral form

$$V(t) = V(0) + \sum_{i=0}^{n} \int_0^t \theta_i(u) dS_i(u), \quad 0 \leq t \leq T. \tag{2.12}$$

Definition 2.9: Self-financing portfolio

A portfolio process $\{\theta(t)\}$ is said to be *self-financing* if the time t portfolio value $V(t)$ is represented by (2.12).

[7]Refer to sub-section 2.1.1 for the other assumptions about the securities market.

Definition 2.10: Replicating portfolio

A contingent claim X is said to be attainable if there exists some self-financing trading strategy $\{\theta(t),\ t \in [0,T]\}$, called a *replicating portfolio*, such that $V(T) = X$, i.e.

$$X = V(0) + \sum_{i=0}^{n} \int_{0}^{T} \theta_i(t) dS_i(t). \tag{2.13}$$

Definition 2.11: Arbitrage opportunity.

An *arbitrage-opportunity* is the existence of some self-financing trading strategy $\{\theta(t),\ t \in [0,T]\}$, such that

1. $V(0) = 0$.
2. $V(T) \geq 0$.
3. $V(T) > 0$ with positive probability .

Remark: The above definition just implies that an arbitrage opportunity is a risk-free way of making profit.

Theorem 2.2. No-arbitrage pricing.

For a given contingent claim X suppose that there exist a replication trading strategy $\{\theta(t),\ t \in [0,T]\}$ as in (2.13). If there are no-arbitrage opportunities in the market, then $V(0)$ is the correct price for the contingent claim X.

Remark: Theorem 2.2 suggests that we determine the initial cost $V(0)$ of the replicating portfolio $\{\theta(t)\}$ given by (2.13) in order to price the contingent claim X under the assumption of no-arbitrage opportunities. (See Shreve([28]), Kijima ([14]),Kwok ([18]), Etheridge ([8]) , Musiela et.al ([21]) , Ross ([25]) and Neftci ([23])).

2.6.1 Risk-Neutral probability measure.

Consider the denominated prices $\widetilde{S}_i(t)$ with $S_0(t)$ being the numeraire i.e. $\widetilde{S}_i(t) = \frac{S_i(t)}{S_0(t)}$. A *numeraire* by definition (See [1]), is any asset price process $\{X(t)\}$ almost surely positive for each $t \in [0,T]$, that is $X(t) > 0$ for all times of t.

Now suppose there exists a probability measure $\widetilde{P}$ under which $\{\widetilde{S}_i(t)\}$ are martingales i.e.

$$\widetilde{E}_t[\widetilde{S}_i(t)] = \widetilde{S}_i(t), \quad 0 \leq t \leq T, \tag{2.14}$$

where $\widetilde{E}_t$ is the conditional expectation under the new probability measure $\widetilde{P}$. Such a measure is known as *martingale measure*. Then the correct price $V(0)$ for a contingent claim X is given by

$$V(0) = \widetilde{E}_t[\widetilde{V}(T)] = \widetilde{E}_t\left[\frac{X}{S_0(T)}\right], \quad 0 \le t \le T, \tag{2.15}$$

where $\widetilde{E} = \widetilde{E}_0$ under some regularity condition.

Definition 2.12: Risk-Neutral Probability Measure.

Given a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$, a probability measure P^* is said to be *risk-neutral* if

1. P^* is equivalent to P, i.e. $P(A) > 0$ if and only if $P^*(A) > 0, \quad \forall\ A \in \mathcal{F}$.
2. $E_t^*[S_i^*(t)] = S_i^*(t), \quad 0 \le t \le T$, holds for all i and t with $S_0(t) = B(t)$, the money market account.

Consider the time t price of a European contingent claim denoted $C(t)$, written on stock $S(t)$ with payoff $h(S(T))$ at maturity time T. Suppose that the contingent claim is replicated through a self-financing strategy, hence the denominated price process $\{C^*(t)\} = \frac{\{C(t)\}}{\{B(t)\}}$ is always martingale under the risk-neutral probability measure. Hence the time t price $C(t)$ of the contingent claim is given by

$$C(t) = B(t)E_t^*\left[\frac{h(S(T))}{B(T)}\right], \quad 0 \le t \le T, \tag{2.16}$$

that is

$$\begin{aligned} C^*(t) &= E_t^*\left[\frac{h(S(T))}{B(T)}\right] \\ \frac{C(t)}{B(t)} &= E_t^*\left[\frac{h(S(T))}{B(T)}\right] \\ C(t) &= B(t)E_t^*\left[\frac{h(S(T))}{B(T)}\right], \quad 0 \le t \le T. \end{aligned}$$

For example a European call option with strike price K , will have the payoff function $h(S(T)) = \{S(T) - K\}_+$.

Theorem 2.3 Fundamental theorem of asset pricing[8]

There are no-arbitrage opportunities if an only if there exists a risk-neutral probability measure. Thus the price of an attainable contingent claim X is given by (2.15) with $S_0(t) = B(t)$ for every replicating portfolio strategy.

Definition 2.13: Complete market.

A securities market is said to be *complete* if every contingent claim is attainable; otherwise, the market is said to be *incomplete*.

Theorem 2.4. Fundamental theorem of asset pricing (stronger version)

A securities market is complete if and only if there exists a unique risk-neutral probability measure.

Now having outline the arbitrage-free pricing of contingent claims, it remains to calculate the expectation given in equation (2.16). To achieve this we use an important result constructed by (Kijima, [14]). This result can then be used to evaluate the expectation in the Black-Scholes setting, which is used for pricing European contingent claims such as call options. Before we state and prove this result, the lemma stated below will be useful in the proof of the proposition that follows.

Lemma: *Suppose the random variable $X \sim N(-\frac{1}{2}\psi^2, \psi^2)$, and its density function is given by $f_X(x)$. Let Y be a random variable defined by the following density*

$$f_Y(x) = e^x f_X(x), \quad x \in \mathbb{R}.$$

Then $f_Y(x)$ defines the normal distribution function with mean $\frac{1}{2}\psi^2$ and variance ψ^2.

Proof

Note that

$$\int_{-\infty}^{\infty} f_Y(x)dy = \int_{-\infty}^{\infty} e^x f_X(x)dx = E[e^X] = M_X(1) = e^{-\frac{1}{2}\psi^2 + \frac{1}{2}\psi^2} = 1,$$

which implies that $f_Y(x)$ is a density.

Now, we show that the density $f_Y(x)$ is normally distributed with mean $\frac{1}{2}\psi^2$ and variance ψ^2. Using moment generating functions (mgf), it follows that

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} e^y f_X(y)dy = \int_{-\infty}^{\infty} e^{y(t+1)} f_X(y)dy = M_X(t+1).$$

[8] See [18],[23],[25],[14],[28].

Therefore

$$\begin{aligned}
M_Y(t) &= M_X(t+1) \\
&= e^{-\frac{1}{2}\psi^2(t+1)+\frac{1}{2}\psi^2(t+1)^2} \\
&= e^{-\frac{1}{2}\psi^2 t-\frac{1}{2}\psi^2+\frac{1}{2}\psi^2 t^2+\psi^2 t+\frac{1}{2}\psi^2} \\
M_Y(t) &= e^{\frac{1}{2}\psi^2 t+\frac{1}{2}\psi^2 t^2},
\end{aligned}$$

and thus indeed $f_Y(x) \sim N(\frac{1}{2}\psi^2, \psi^2)$. ■

Now we are in a position to state and prove the following result, which is the key to the evaluation of the expectation in equation (2.16) and it is given in the following proposition.

Proposition 2.6 *Let $\sigma(t)$ be a deterministic function of time t such that $\int_0^T \sigma^2(t)dt < \infty$. Suppose that the price process $\{S(t)\}$ follows the SDE*

$$dS(t) = \sigma(t)S(t)dz, \quad t \geq 0,$$

where $\{z(t)\}$ is a standard Brownian motion. Then

$$S(t) = S\, exp\Big\{-\frac{1}{2}\psi^2 + \int_0^t \sigma(u)dz(u)\Big\},$$

where $\psi^2 = \int_0^t \sigma^2(u)du$, and

$$E\Big[\{S(t) - K\}_+\Big] = S\Phi(d) - K\Phi(d-\psi),$$

where $S(0) = S$, $d = \frac{log\left(\frac{S}{K}\right)+\frac{1}{2}\psi^2}{\psi}$, $\psi > 0$ and $\Phi(z)$ is the distribution function of the standard normal distribution.

Proof

The SDE $dS(t) = \sigma(t)Sdz$, is similar to that of the security price process given in equation (2.9), but without the drift term. Applying Ito's formula (Theorem 2.1) , we construct the log-price of $S(t)$ as follows

$$dlog[S(t)] = [f_t(x,t) + f_x(x,t)\mu(x,t) + \frac{1}{2}f_{xx}(x,t)\sigma^2(x,t)]dt + [f_x(x,t)\sigma(x,t)]dz.$$

Substituting $f_t(x,t) = 0$, $f_x(x,t) = \frac{1}{x} = \frac{1}{S(t)}$, $f_{xx}(x,t) = -\frac{1}{x^2} = -\frac{1}{S^2(t)}$, $\mu(x,t) = 0$ and $\sigma(x,t) = \sigma(t)S(t)$, when $f(x) = log(x)$ with $x = S(t)$ it follows that

$$\begin{aligned} dlog[S(t)] &= [0 + (\frac{1}{S(t)})(0) + \frac{1}{2}(-\frac{1}{S^2(t)})\sigma^2(t)S^2(t)]dt + [(\frac{1}{S(t)})\sigma(t)S(t)]dz \\ dlog[S(t)] &= -\frac{1}{2}\sigma^2(t)dt + \sigma(t)dz. \end{aligned}$$

Integrating both sides, it follows that

$$\begin{aligned} log[S(t)] - log[S] &= -\frac{1}{2}\int_0^t \sigma^2(u)du + \int_0^t \sigma(u)dz(u) \\ log\left[\frac{S(t)}{S}\right] &= -\frac{1}{2}\int_0^t \sigma^2(u)du + \int_0^t \sigma(u)dz(u), \end{aligned}$$

which implies that

$$S(t) = S\,exp\left\{-\frac{1}{2}\psi^2 + \int_0^t \sigma(u)dz(u)\right\}, \quad \text{where } \psi^2 = \int_0^t \sigma^2(u)du,$$

with $S(0) = S$ as required. Now let the process $Y(t)$ be defined by $Y(t) = -\frac{1}{2}\psi^2 + \int_0^t \sigma(u)dz(u)$. It follows that $S(t) = Se^{Y(t)}$, which implies that the price process $Y(t) = log\left(\frac{S(t)}{S}\right)$, is normally distributed.

Thus the mean of $Y(t) = Y$ (for simplicity) is given by

$$E(Y) = E\left[-\frac{1}{2}\psi^2 + \int_0^t \sigma(u)dz(u)\right] = -\frac{1}{2}\psi^2 + E\left[\int \sigma(u)dz(u)\right] = -\frac{1}{2}\psi^2.$$

This follows from property (3) and (5) of the Ito integral[9].

The variance is given by

[9] See Section 2.5.2.

$$\begin{aligned}
Var(Y) &= E[Y - E(Y)]^2 \\
&= E\Big[-\frac{1}{2}\psi^2 + \int_0^t \sigma(u)dz(u) - (-\frac{1}{2}\psi^2)\Big]^2 \\
&= E[\int_0^t \sigma(u)dz(u)]^2 \\
&= E[\int_0^t \sigma^2(u)du] \\
&= \int_0^t \sigma^2(u)du \\
Var(Y) &= \psi^2.
\end{aligned}$$

In the above calculation of the variance we used property (4) (Ito isometry) of the Ito integral.[10]. Thus $Y \sim N(-\frac{1}{2}\psi^2, \psi^2)$ and the moment generating function of Y is given by

$$M_Y(t) = E[e^{tY}] = exp[-\frac{1}{2}\psi^2 t + \psi^2 t^2],$$

which implies that

$$E[e^Y] = M_Y(1) = exp[-\frac{1}{2}\psi^2 + \psi^2].$$

Now the next task is to prove that the expected value of the payoff is given by

$$E\Big[\{S(t) - K\}_+\Big] = S\Phi(d) - K\Phi(d - \psi).$$

First we consider the probability $S(t) \geq K$, that is exercising a European call option, then we write the expectation as follows

$$E\Big[\{S(t) - K\}_+\Big] \quad = \quad E\Big[(Se^Y - K)\, 1_{\{Se^Y \geq K\}}\Big],$$

where $S(t) = Se^Y$ and $1_{\{Se^Y \geq K\}} = \begin{cases} 1 & Se^Y \geq K \\ 0 & \text{elsewhere} \end{cases}$, then

$$\begin{aligned}
E\Big[\{S(t) - K\}_+\Big] &= E\Big[Se^Y 1_{\{Se^Y \geq K\}}\Big] - E\Big[K 1_{\{Se^Y \geq K\}}\Big] \\
E\Big[\{S(t) - K\}_+\Big] &= SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] - KE\Big[1_{\{Se^Y \geq K\}}\Big]. \qquad (2.17)
\end{aligned}$$

Next we want to evaluate the right hand side (RHS) of equation (2.17). We start by evaluating the second term on the RHS i.e.

[10]See Section 2.5.2.

$$\begin{aligned} KE\Big[1_{\{Se^Y > K\}}\Big] &= KP\Big[Se^Y \geq K\Big] \\ &= KP\Big[e^Y \geq \frac{K}{S}\Big] \\ &= KP\Big[Y \geq log\Big(\frac{K}{S}\Big)\Big], \end{aligned}$$

but we know that $Y \sim N(-\frac{1}{2}\psi^2, \psi^2)$, thus standardizing yields

$$\begin{aligned} KE\Big[1_{\{Se^Y > K\}}\Big] &= KP\Bigg[\frac{Y - (-\frac{1}{2}\psi^2)}{\psi} \geq \frac{log\Big(\frac{K}{S}\Big) - (-\frac{1}{2}\psi^2)}{\psi}\Bigg] \\ &= KP\Bigg[Z \geq \frac{log\Big(\frac{K}{S}\Big) + \frac{1}{2}\psi^2}{\psi}\Bigg] \\ &= K\Phi\Bigg[\frac{-log\Big(\frac{K}{S}\Big) - \frac{1}{2}\psi^2}{\psi}\Bigg] \\ &= K\Phi\Bigg[\frac{log\Big(\frac{S}{K}\Big) - \frac{1}{2}\psi^2}{\psi}\Bigg]. \end{aligned}$$

Therefore

$$KE\Big[1_{\{Se^Y > K\}}\Big] = K\Phi\Bigg[\frac{log\Big(\frac{S}{K}\Big) - \frac{1}{2}\psi^2}{\psi}\Bigg] = K\Phi(d - \psi), \tag{2.18}$$

where $d = \frac{log\left(\frac{S}{K}\right) + \frac{1}{2}\psi^2}{\psi}$, $\psi > 0$.

Next we solve the first term on the RHS of equation (2.17) i.e $SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big]$. Now using lemma 2.1, if we let $Y \sim N(-\frac{1}{2}\psi^2, \psi^2)$, with density function $f_Y(y)$, then the random variable V, defined by the density function,

$$f_V(y) = e^y f_Y(y), \quad y \in \mathbb{R},$$

is normally distributed with mean $\frac{1}{2}\psi^2$ and variance ψ^2 i.e. $V \sim N(\frac{1}{2}\psi^2, \psi^2)$ and $f_Y(y) = e^{-y} f_V(y)$, therefore

$$
\begin{aligned}
SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] &= S\int_{-\infty}^{\infty} e^y 1_{\{Se^y \geq K\}} f_Y(y)dy \\
&= S\int_{-\infty}^{\infty} e^y 1_{\{Se^y \geq K\}} e^{-y} f_V(y)dy \\
&= S\int_{-\infty}^{\infty} 1_{\{Se^y \geq K\}} f_V(y)dy \\
& \quad S\int_{-\infty}^{\infty} 1_{\{Se^v \geq K\}} f_V(v)dv \\
&= SE\Big[1_{\{Se^V \geq K\}}\Big].
\end{aligned}
$$

It then follows that,

$$
\begin{aligned}
SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] &= SE\Big[1_{\{Se^V \geq K\}}\Big] \\
&= SP\Big[Se^V \geq K\Big] \\
&= SP\Big[V \geq log\Big(\frac{K}{S}\Big)\Big],
\end{aligned}
$$

but $V \sim N(\frac{1}{2}\psi^2, \psi^2)$, thus standardizing yields

$$
\begin{aligned}
SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] &= SP\left[\frac{V - \frac{1}{2}\psi^2}{\psi} \geq \frac{log\Big(\frac{K}{S}\Big) - \frac{1}{2}\psi^2)}{\psi}\right] \\
&= SP\left[Z \geq \frac{log\Big(\frac{K}{S}\Big) - \frac{1}{2}\psi^2}{\psi}\right] \\
&= S\Phi\left[\frac{-log\Big(\frac{K}{S}\Big) + \frac{1}{2}\psi^2}{\psi}\right] \\
&= S\Phi\left[\frac{log\Big(\frac{S}{K}\Big) + \frac{1}{2}\psi^2}{\psi}\right].
\end{aligned}
$$

Therefore

$$
SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] = S\Phi\left[\frac{log\Big(\frac{S}{K}\Big) + \frac{1}{2}\psi^2}{\psi}\right] = S\Phi(d),
$$

where $d = \frac{log\big(\frac{S}{K}\big)+\frac{1}{2}\psi^2}{\psi},\ \psi > 0.$

Combining the two expressions yields the desired result that

$$E\Big[\{S(t)-K\}_+\Big] = SE\Big[e^Y 1_{\{Se^Y \geq K\}}\Big] - KE\Big[1_{\{Se^Y \geq K\}}\Big] = S\Phi(d) - K\Phi(d-\psi),$$

thus proving Proposition 2.6. ■

2.6.2 Forward-Neutral probability measure.

Another type of martingale measure, which takes into account a stochastic interest-rate economy is called the forward-neutral evaluation method. In particular for this thesis we use the forward-neutral probability measure for the arbitrage-free pricing of swaptions and caplets. Forward-neutral probability measure differs from the risk-neutral probability measure because of two main reasons. Firstly, there is a difference in the choice of numeraire. For the risk-neutral measure the choice of numeraire is the money market account $B(t)$, and for the forward-neutral measure the choice of numeraire is the time t default-free discount bond which matures at time T, denoted by $v(t,T)$. It is important to note that we assume that at maturity time T , the value of the discount bond is unity i.e. $v(T,T) = 1$. Secondly the risk-neutral method requires the joint distribution of the payoff function $h(S(T))$ and $B(T)$ in order to evaluate the expectation (2.16), whereas the forward-neutral method only requires the marginal distribution of $h(S(T))$.

Now consider the denominated prices of $S_i(t)$ with $v(t,T)$ being the numeraire i.e. $S_i^T(t) = \frac{S_i(t)}{v(t,T)}$, $t \in [0,T]$. Suppose there exists a probability measure P^T under which $\{S^T(t)\}$ is a martingale i.e.

$$E_t^T[S_i^T(T)] = S_i^T(t), \quad 0 \leq t \leq T, \tag{2.19}$$

where E_t^T is the conditional expectation under the new probability measure P^T, called the forward-neutral probability measure, given $\mathcal{F}_t$. Then the correct price $V(0)$ for a contingent claim X is given by

$$V(0) = E^T[V^T(T)] = v(0,T)E^T[X], \quad 0 \leq t \leq T, \tag{2.20}$$

where $E^T = E_0^T$ under some regularity condition.

Definition 2.13: Forward-Neutral Probability Measure.

Given a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$, a probability measure P^T is said to be *forward-neutral* if

1. P^T is equivalent to P, i.e. $P(A) > 0$ if and only if $P^T(A) > 0 \ \ \forall\ A \in \mathcal{F}$.
2. $E_t^T[S_i^T(T)] = S_i^T(t), \quad 0 \leq t \leq T$, holds $\forall\ i$ and t with $S_0(t) = v(t,T)$, the default-free discount bond.

Now consider a European contingent claim with time t price denoted by $C(t)$. Then according to the forward-neutral method the denominated price process $\{C^T(t)\} = \frac{\{C(t)\}}{\{v(t,T)\}}$, is a martingale under the forward-neutral probability measure. Hence the time t price of the contingent claim is given by

$$C(t) = v(t,T)E_t^T\Big[h(S(T))\Big], \tag{2.21}$$

that is

$$\begin{aligned} C^T(t) &= E_t^T\Big[\frac{h(S(T))}{v(T,T)}\Big] \\ \frac{C(t)}{v(t,T)} &= E_t^T\big[h(S(T))\big], \text{ since } v(T,T) = 1. \\ C(t) &= v(t,T)E_t^T\big[h(S(T))\big]. \end{aligned}$$

The expected value under the forward neutral measure, that is, $E_t^T\Big[h(S(T))\Big]$, can be evaluated using Proposition 2.6. Having discussed the risk-neutral and forward-neutral valuation methods, the following section discusses the change of measure method, which enables use to change from one equivalent probability measure to another probability measure, for example from the empirical probability P to the risk-neutral probability $P^*$and from P^* to the forward neutral probability measure P^T in the continuous-time framework.

2.7 Change of measure and Girsanov's Theorem

In the discrete-time framework once we have found a martingale measure, contingent claim pricing is reduced to calculating the expectations under that measure. In contrast, in the continuous-time framework we can not find a martingale measure using linear algebra. The effective evaluation of contingent claims in the continuous-time framework often requires the transformation of the underlying price process with drift into a martingale, but under a new probability measure, which we shall denote as $\widetilde{P}$ [11]. This transformation can be performed effectively using Girsanov's theorem, and the Radon-Nikodym derivative, which relates the transformation between the two measures ([18]) . In this section we define the the Radon-Nikodym derivative as done by Shreve ([28]), and state without proof[12] the Girsanov's theorem as stated by Kwok ([18]).

[11]Note that $\widetilde{P}$ is used to define the general case of a new probability measure, which can be viewed as the risk or forward neutral probability measure.

[12]Refer to Shreve ([28]) for the one dimensional proof of Girsanov's theorem.

2.7.1 The Radon-Nikodym derivative

As mentioned earlier no-arbitrage valuation of financial derivatives can be done in the risk-neutral and forward-neutral probability measure. This means that we can change probability measures from the empirical probability measure P to an equivalent probability measure $\widetilde{P}$. One then wonders what tools are required to change from one measure to another. Such a tool is known as the *Radon-Nikodym derivative.* To illustrate this, consider a standard normal random variable X defined on $(\Omega, \mathcal{F}, P)$. Now consider another random variable Y such that $Y = X + \beta$, where β is a positive constant i.e $\beta > 0$, then $Y \sim N(\beta, 1)$ under P. Now suppose we want to change to a new probability measure $\widetilde{P}$ on Ω under which Y is a standard normal variable, without simply subtracting β from Y. This can be achieved by introducing a variable $\omega \in \Omega$, such that we assign less probability to those values for which $Y(\omega)$ is sufficiently positive and more to those ω for which $Y(\omega)$ is negative. Thus we want to change the distribution of Y without changing the random variable Y (Shreve, [28]). Similarly in financial context, when we can change from empirical probability P to the risk-neutral probability measure P^*, by changing the distribution of the security prices, without changing the security prices themselves. In order to achieve this, we define a random variable

$$\eta(\omega) = e^{-\beta X(\omega) - \frac{1}{2}\beta^2}, \quad \text{for all } \omega \in \Omega,$$

which serves as a Radon-Nikodym derivative, for changing from one probability measure to another equivalent probability measure. This random variable has two important properties listed below.

(i) $\eta(\omega) > 0$, for all $\omega \in \Omega$.

(ii) $E[\eta] = 1$.

Here it is important to note that in context to the theory of security price processes discussed in earlier sections, the standard normal random variable X is replaced with the standard Brownian motion $\{z(t)\}$ which is also standard normal. We can then use this random variable to create a new probability measure $\widetilde{P}$ by adjusting the probabilities of the events in Ω. This is summarized in the theorem below.

Theorem 2.5 (Radon-Nikodym)

Let P and $\widetilde{P}$ be equivalent probability measures defined on probability space $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable η such that $E[\eta] = 1$ and

$$\widetilde{P}(A) = E(\eta 1_A) = \int_A \eta(\omega) dP(\omega), \quad \textit{for every } A \in \mathcal{F} \textit{ and } \omega \in \Omega, \tag{2.22}$$

where 1_A denotes the indicator function of event A.

To simplify the expression of the Radon-Nikodym derivative shown above, we can rewrite it in terms of the ratios of the probability measures and it is given in the definition below.

Definition 2.14: Radon-Nikodym derivative

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\widetilde{P}$ be another probability measure on $(\Omega, \mathcal{F})$, that is equivalent to P, and let η be an almost surely positive random variable that relates P to $\widetilde{P}$ via equation (2.22). Then η is called the Radon-Nikodym derivative of $\widetilde{P}$ with respect to P, and can be written as follows

$$\eta = \frac{d\widetilde{P}}{dP}. \tag{2.23}$$

Now in what follows, we show that choosing the random variable $\eta(\omega) = e^{-\beta X(\omega) - \frac{1}{2}\beta^2}$ leads to $Y = X + \beta \sim N(0,1)$ under the new probability measure $\widetilde{P}$ as stated earlier.

Note that

$$\begin{aligned}
\widetilde{P}(Y \le a) &= \int_{\{\omega; Y(\omega) \le a\}} \eta(\omega) dP(\omega) = \int_{\Omega} 1_{\{Y(\omega) \le a\}} \eta(\omega) dP(\omega) \\
&= \int_{\Omega} 1_{\{X(\omega)+\beta \le a\}} \eta(\omega) dP(\omega) = \int_{\Omega} 1_{\{X(\omega) \le a-\beta\}} \eta(\omega) dP(\omega) \\
&= \int_{-\infty}^{\infty} 1_{\{x \le a-\beta\}} e^{-\beta x - \frac{1}{2}\beta^2} \frac{1}{2\pi} e^{-\frac{1}{2}x^2} dx \quad \text{where } X \sim N(0,1) \text{ under } P \\
&= \int_{-\infty}^{a-\beta} \frac{1}{2\pi} e^{-\beta x - \frac{1}{2}\beta^2 - \frac{1}{2}x^2} dx = \int_{-\infty}^{a-\beta} \frac{1}{2\pi} e^{-\frac{1}{2}(x+\beta)^2} dx = \int_{-\infty}^{a} \frac{1}{2\pi} e^{-\frac{1}{2}y^2} dy.
\end{aligned}$$

Therefore Y is indeed a standard normal random variable under the new probability measure $\widetilde{P}$ as claimed.

In order to price and hedge in the Black-Scholes framework we require two fundamental results. The first will allow us to change probability measure so that the discounted securities prices are martingales (Etheridge, [8]). This is the Radon Nikodym derivative which we have discussed in the section above. The second fundamental result is known as Girsanov's theorem which is a useful tool to determine an equivalent martingale measure. Girsanov's theorem is stated below without proof.

Theorem 2.6. (Girsanov's Theorem)

Let $\{z_P(t)\}$ be a standard Brownian process under the probability measure P. Let $\{\mathcal{F}_t,\ t \geq 0\}$, be the filtration generated by $\{z_P(t)\}$. Consider a $\mathcal{F}_t$-adapted stochastic process $\{\beta(t)\}$ which satisfies the Novikov condition[13]

$$E\Big[exp\{\frac{1}{2}\int_0^t \beta^2(s)\,ds\}\Big] < \infty,$$

and also consider the measure $\widetilde{P}$ such that the Radon-Nikodym derivative is defined by

$$\frac{d\widetilde{P}}{dP} = \eta(t),$$

where

$$\eta(t) = exp\Big(\int_0^t -\beta(s)dz - \frac{1}{2}\int_0^t \beta^2(s)ds\Big). \tag{2.24}$$

Then under the probability measure $\widetilde{P}$, the Ito process

$$z_{\widetilde{P}}(t) = z_P(t) + \int_0^t \beta(s)\,ds, \tag{2.25}$$

is standard Brownian motion.

Having outlined the change of measure technique using the Radon-Nikodym derivation and Girsanov's theorem, we are now in a position to discuss the application of these techniques for changing from one numeraire to another.

2.7.2 Change of numeraire

The change of numeraire technique is seen to be a powerful tool for analytical pricing of financial derivatives. For example suppose the contingent claim X is attainable as shown in equation (2.13). Then comparing the risk-neutral and forward-neutral martingale measures, it then follows from equations (2.15) with $S_0(t) = B(t)$ and (2.20) , that the correct price $V(0)$ for a contingent claim X is given by

$$V(0) = E^*\left[\frac{X}{B(T)}\right] = v(0,T)E^T[X].$$

[13]This condition implies that $\beta(t)$ cannot increase or decrease "too fast" over time.

In order to see the link between these two measures, we take $X = 1_A$ for all $A \in \mathcal{F}$, such that

$$\begin{aligned} V(0) &= E^*\left[\frac{1_A}{B(T)}\right] = v(0,T)E^T[1_A] \\ &= E^*\left[\frac{1_A}{B(T)}\right] = v(0,T)P^T(A), \end{aligned}$$

which implies that

$$P^T(A) = E^*\left[\frac{1_A}{v(0,T)B(T)}\right],$$

but from the definition of the Radon-Nikodym derivative we know that $\widetilde{P}(A) = E(\eta 1_A)$ and $\eta = \frac{d\widetilde{P}}{dP}$, thus in this example it implies that

$$P^T(A) = E^*\left[\eta 1_A\right], \quad \text{where } \eta = \frac{1}{v(0,T)B(T)} = \frac{dP^T}{dP^*},$$

and noting that $v(T,T) = 1$ and $B(0) = 1$, therefore the Radon-Nikodym that effects change of numeraire from the P^* to P^T is given by

$$\frac{dP^T}{dP^*} = \frac{v(T,T)/v(0,T)}{B(T)/B(0)}.$$

In general the Radon-Nikodym derivative that effects change of measure from the numeraire-measure pair $(N(t), P^N)$ to the other pair $(M(t), P^M)$ is given by

$$\frac{dP^N}{dP^M}|_{\mathcal{F}_t} = \frac{N(t)/N(0)}{M(t)/M(0)}, \quad t \in [0,T],$$

where $\frac{dP^N}{dP^M}|_{\mathcal{F}_t}$, means that the Radon-Nikodym derivative is a process depending on the information $\mathcal{F}_t$ available on the ratio of numeraire-measure pairs $(N(t), P^N)$ and $(M(t), P^M)$ up to time $t \leq T$.

Now that we have discussed the basic tools for pricing financial derivatives, in the proceeding section we shall outline the Black-Scholes model for pricing European contingent claims in particular call options.

2.8 The Black-Scholes Model.

Black and Scholes ([2]) tackled the problem of pricing and hedging a European call option on a non-dividend paying stock. The model suggested by Black and Scholes describes the behavior of prices in a continuous-time framework with one risky asset (a stock with price $S_1(t) = S(t)$ at time t) and one risk-less asset (with price $S_0(t)$ at time t) (Lamberton and Lapeyre, [19]). In this section we want to show that martingale pricing theory gives the fair price of a European call option as the expectation of the discounted terminal payoff under equivalent martingale measure. Under the risk-neutral measure, the risk-free security is the money money market account, $B(t)$ and under the forward-neutral probability measure, it is the default-free discount bond, $v(t,T)$. Using Proposition 2.6 we can determine the Black-Scholes formula for a call option. First we need to determine the SDE of the risky stock price under the risk-neutral probability measure P^*, then we can apply Proposition 2.6 to evaluate the expected future payoff. Before we proceed using the Black-Scholes model, it is required of us to make some basic assumptions about the securities market so that it is mathematically viable to implement the model. These assumptions have been summarized below with reference to Black and Scholes ([2]).

Assumptions: The Black-Scholes model

1. The option written on the security $\{S(t)\}$ is of "European" type, i.e. it can only be exercised at maturity.
2. Security price processes $\{S(t)\}$ follow the geometric Brownian motion with drift $\mu(S,t)$ and volatility $\sigma(S,t)$.
3. The underlying security pays no dividends before the option matures.
4. There are no arbitrage opportunities in the market.
5. The risk-less interest rate r is known and constant over time.
6. Security trading is continuous, there are no transaction costs when buying or selling stock and no penalties for short selling .

Before we proceed to state the Black-Scholes formula for pricing call options we first need to show that there exists a probability measure equivalent to P under which the discounted stock price $S^*(t) = \frac{S(t)}{S_0(t)}$ are martingales. Note that the risk free security $S_0(t) = B(t)$, is the money market account. The following construction is based on the construction done by Lamberton et.al ([19]) and Kijima ([14]). Let $r(t)$ be the time t instantaneous spot rate, and assume that $\{r(t)\}$ is a non-negative process, adapted to filtration $\{\mathcal{F}_t\}$.

Now if the interest rate is continuous compounding, then the time t money market account $B(t)$ is defined by

$$B(t) = e^{\int_0^t r(u)du},$$

which can be expressed in the differential form

$$dB(t) = r(t)B(t)dt.$$

On the other hand consider the risky security defined by equation (2.9) i.e.

$$dS = \mu(S,t)Sdt + \sigma(S,t)Sdz.$$

Then let us define the process $\{z^*(t)\}$ by

$$z^*(t) = z(t) + \frac{\mu(S,t) - r(t)}{\sigma(S,t)}. \tag{2.26}$$

Let P^* be the probability measure that makes $\{z^*(t)\}$ a standard Brownian motion. The existence of such a probability measure is guaranteed by Girsanov's theorem (Theorem 2.6), where $\beta(t) = \frac{\mu(S,t)-r(t)}{\sigma(S,t)}$, is known as the market price of risk. Thus the SDE of the risky stock $S(t)$ under P^* is given by

$$\begin{aligned}
dS(t) &= \mu(S,t)Sdt + \sigma(S,t)S\Big[dz^* - \frac{\mu(S,t) - r(t)}{\sigma(S,t)}dt\Big] \\
&= \mu(S,t)Sdt + \sigma(S,t)Sdz^* - S[\mu(S,t) - r(t)]dt \\
dS(t) &= Sr(t)dt + \sigma(S,t)Sdz^*, \qquad (2.27)
\end{aligned}$$

where $\{z^*(t)\}$ a standard Brownian motion under P^*. This means that the mean return of stock $S(t)$ under P^* is equal to that of the risk free security $B(t)$, hence under P^* the two securities have the same mean rate of return while volatilities are different. Now in the Black-Scholes settings, where the risk free interest rate r and the volatility σ are positive constants we obtain from equation (2.27) that

$$dS(t) = Srdt + \sigma Sdz^*.$$

It follows from Ito's formula that

$$dlogS(t) = \Big(r - \frac{1}{2}\sigma^2\Big)dt + \sigma dz^*,$$

which implies that

$$S(t) = S\,exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma z^*(t)\right], \quad 0 \le t \le T, \tag{2.28}$$

with $S(0) = S$. Hence the stock price $S(t)$ is log-normally distributed, with mean $\left(r - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$, under P^*. Next we show that in the Black-Scholes setting, the discounted price process $\{S^*(t)\}$ is martingale, that is the SDE for the discounted price has zero drift. To achieve we use the division rule (Proposition 2.5) to determine the SDE for the discounted price as follows .

Let

$$\frac{dX}{X} = \frac{dS(t)}{S} = rdt + \sigma dz^*,$$

which implies that $\mu_X(t) = r$ and $\sigma_X(t) = \sigma$. Also we let

$$\frac{dY}{Y} = \frac{dB(t)}{B(t)} = rdt,$$

which implies that $\mu_Y(t) = r$ and $\sigma_Y(t) = 0$. Therefore

$$\begin{aligned}
\frac{dD}{D} &= \frac{dS^*(t)}{S^*} = \Big[\mu_X(t) - \mu_Y(t) - \sigma_Y(t)(\sigma_X(t) - \sigma_Y(t))\Big]dt + [\sigma_X(t) - \sigma_Y(t)]dz^* \\
&= [r - r - 0(\sigma - 0)]dt + [\sigma - 0]dz^* \\
&= \sigma dz^*.
\end{aligned}$$

Therefore the discounted price process $\{S^*(t)\}$ is a solution to the SDE

$$dS^*(t) \quad = \quad \sigma S^* dz^*. \tag{2.29}$$

Thus the price process $\{S^*(t)\}$ is martingale under P^* since the drift term is equal to zero. Given this SDE we are in a position to use Proposition 2.6 to derive the Black-Scholes formula for a European call option.

Theorem 2.7: Black-Scholes formula for a European call option.

Consider a European call option written on stock $S(t)$ with payoff function $\{S(T) - K\}_+$, where K is the strike price and T is the maturity date. Then the time t value of the call option (premium) ,under the risk neutral probability measure P^* is given by

$$c(S,t) = S(t)\Phi(d) - Ke^{-r(T-t)}\Phi(d - \sigma\sqrt{T-t}), \tag{2.30}$$

where

$$d = \frac{log\left(\frac{S(t)}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{2.31}$$

and σ denotes the implied volatility of the stock $S(t)$ and $\Phi(z)$ is the distribution function of the standard normal distribution i.e. $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Proof

Given the SDE of the discounted price process $\{S^*(t)\}$, i.e. $dS^*(t) = \sigma S^* dz^*$, it follows from Proposition 2.6 that

$$S^*(T) = S^* \, exp\left\{-\frac{1}{2}\psi^2 + \sigma z^*(T)\right\}, \quad 0 \le t \le T,$$

where $\psi^2 = \sigma^2 T$ and $S(0) = S^*$. As before if we define a random variable $Y = -\frac{1}{2}\psi^2 + \sigma z^*(T)$, it implies that $S^*(T) = S^* e^Y$, such that $Y = log\left[\frac{S^*(T)}{S^*}\right] \sim N(-\frac{1}{2}\psi^2, \psi^2)$.

Now assuming that the call option is replicated through some self-financing strategy, then the time $t = 0$ price of the call option $c(S,0)$ with reference to equation(2.16) where $h(S(T)) = \{S(T) - K\}_+$ and $B(t) = e^{rt}$ (risk-free money market account) is given by

$$c^*(S,0) = E^*\left[\frac{\{S(T) - K\}_+}{B(T)}\right],$$

which implies that

$$c(S,0) \quad = \quad E^*\left[\{S^*(T) - e^{-rT}K\}_+\right].$$

Now it follows from Proposition 2.6 that

$$
\begin{aligned}
c(S,0) &= E^*\Big[(S^*e^Y - e^{-rT}K)\,1_{\{S^*e^Y \geq K\}}\Big] \\
&= S^*E\Big[e^Y 1_{\{S^*e^Y \geq K\}}\Big] - e^{-rT}KE\Big[1_{\{S^*e^Y \geq K\}}\Big] \\
&= S^*\Phi\left[\frac{log\Big(\frac{S^*}{K}\Big) + \frac{1}{2}\psi^2}{\psi}\right] - e^{-rT}K\Phi\left[\frac{log\Big(\frac{S*}{K}\Big) - \frac{1}{2}\psi^2}{\psi}\right].
\end{aligned}
$$

And we also know that under the Black-Scholes framework $\psi^2 = \sigma^2 T$, therefore

$$
\begin{aligned}
c(S,0) &= S^*\Phi\left[\frac{log\Big(\frac{S^*}{K}\Big) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] - e^{-rT}K\Phi\left[\frac{log\Big(\frac{S*}{K}\Big) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] \\
&= S^*\Phi\left[\frac{log\Big(\frac{S^*}{K}\Big) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] - e^{-rT}K\Phi\left[\frac{log\Big(\frac{S*}{K}\Big) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right],
\end{aligned}
$$

but from equation (2.28) it follows that the time $t = 0$ price of the call option is given

$$c(S,0) = S\Phi(d) - e^{-rT}K\Phi(d - \sigma\sqrt{T}),$$

where

$$d = \left[\frac{log\Big(\frac{S}{K}\Big) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right],$$

and therefore the time t value of the call option denoted $c(S,t)$ is given by equation (2.30). ■

Finally now that we have discussed the general outline of the continuous-time model for the securities market, we are fully equipped to introduce the main focus of this thesis which is the "market models" i.e. the LIBOR Market Model (LMM) and Swap Market Model (SMM) used for arbitrage-free pricing of caplets and swaptions respectively.

Chapter 3

LIBOR Market Model

In Chapter 3 we discuss the detailed construction of the LIBOR Market Model (LMM). The idea behind the LMM is to construct an arbitrage-free interest rate model that implies a pricing formula for caplets that has the same structure as the Black pricing formula for a caplet that is used by market practitioners (De Jong, Driessen, & Pelsser, [7]) . The pricing is achieved by utilizing the forward LIBOR rates which are modelled using geometric Brownian motion , that is we assume a log-normal distribution for the forward LIBOR rate. The forward LIBOR rates cannot be traded on the market, that is one cannot buy n units of the LIBOR rates, thus the underlying asset for pricing interest-rate derivatives is the time t price of a family of zero-coupon (default-free discount) bonds, that matures at time $T \leq \tau$ denoted by $v(t,T)$. Musiela and Rutkowski ([21]) define a family of bond prices as an arbitrary family of strictly positive real-valued adapted processes $v(t,T) \quad t \in T$ with $v(T,T) = 1$ for every $T \in [0, \tau]$.

This paper focuses on the formulation of the LMM cited in De Jong et.al ([7]), following the construction done by Musiela and Rutkowski ([22]) and Jamshidan ([12]). Musiela et.al ([22]) and Jamshidan ([12]) suggest that, in order to price and hedge caplets and swaptions it is not necessary that a continuum of bond prices nor a money market account to exist explicitly. On the other hand Brace, Gatarek and Musiela ([4]) construct the LMM based on a continuum of bonds so that it fits the framework of Heath, Jarrow and Morton ([11]) which Jamshidan ([12]) found complicated because of the fact that the LIBOR market model dynamics were specified in the risk-neutral measure and as such still relied on the continuously compounded spot rates. Thus for this thesis the construction of the LMM and SMM - which lead to arbitrage-free pricing of caplets and swaptions respectively - will be implemented under the forward-neutral probability measure.

3.1 Dynamics of the Forward LIBOR Process: LIBOR Market Model

As introduced in Chapter 1, the LMM is used to model forward rate agreements under which the reference forward interest rate used is the London Inter-Bank Offered Rate (LIBOR). LIBOR rates are commonly used in international financial markets and are offered by the commercial banks in London, so that the LIBOR rates should reflect credit risk . This section gives an outline and discusses the dynamics around the forward LIBOR rates which lead to the construction of the LMM

Definition 3.1: LIBOR rate

The LIBOR rate $L(t, t+\delta)$ contracted at time t is the solution to the equation

$$1 + \delta L(t, t+\delta) = \frac{1}{v(t, t+\delta)}, \tag{3.1}$$

where $\delta > 0$ is the time length covered by the LIBOR interest rate.[1]

Consider a fixed set of increasing maturities also known as the tenor structure for the LIBOR rate given by $T_1 < ... < T_i < T_{i+1} < ... < T_N$. Define $\delta_i = T_{i+1} - T_i,\ i = 1, 2, ..., N-1$, where δ_i is the tenor and $\frac{1}{\delta_i}$ is the day count factor. Let $v_i(t) \equiv v(t, T_i)$ denote the time t price of a default-free discount bond maturing at time T_i. Let $L_i(t) \equiv L_i(t, T_i, T_{i+1})$ denote the forward LIBOR rate contracted at time t (where $t \leq T_i$), for the period $[T_i, T_{i+1}]$, that is $L_i(t)$ is reset at dates $T_i,\ i = 1, ..., N-1$ known as the reset dates and is valid for the period $\delta_i = T_{i+1} - T_i$. It follows that equation (3.1) can be written as

$$1 + \delta_i L_i(t) = \frac{v(t, T_i)}{v(t, T_{i+1})} = \frac{v_i(t)}{v_{i+1}(t)}, \quad i = 1, 2, ..., N-1, \tag{3.2}$$

which implies that the T_i forward LIBOR rate at time t is given by

$$L_i(t) = \frac{1}{\delta_i}\left(\frac{v_i(t)}{v_{i+1}(t)} - 1\right), \quad i = 1, 2, ..., N-1, \quad t \geq 0, \tag{3.3}$$

with initial term structure $L_i(0)$.

The forward LIBOR rate in (3.3) is used in the construction of the log-normal LMM which in turn is used to determine a pricing formula for caplets and swaptions, which are similar in structure to that of the Black-Scholes formula . As shown in (3.3) associated with the forward LIBOR rate is the time t price of bond prices with corresponding tenor date $T_i,\ i = 1, 2, ..., N$.

[1] The time length δ is mostly 3 months or 6 months i.e. $\delta = 0.25$ and $\delta = 0,5$ respectively.

Assuming that the bond prices follow Ito processes under the empirical probability measure P, the SDE for the bond prices is given by[2]

$$\frac{dv_i(t)}{v_i(t)} = \mu_i(t)dt + \sigma_i(t)dz(t), \tag{3.4}$$

where $z(t)$ or z in short is a standard Brownian motion. The function $\mu_i(t)$ is the drift function and $\sigma_i(t)$ is the diffusion coefficient or the volatility function for the bond prices.

Applying Ito's division rule (Proposition 2.5) to equation (3.3), the forward LIBOR rate satisfies the following SDE under empirical probability, P,

$$dL_i(t) = \mu_i^L(t)dt + \gamma_i(t)dz, \quad i = 1, 2, ...N-1, \tag{3.5}$$

where

$$\gamma_i(t) = \frac{v_i(t)}{\delta_i v_{i+1}(t)}[\sigma_i(t) - \sigma_{i+1}(t)], \tag{3.6}$$

and

$$\mu_i^L(t) = \frac{v_i(t)}{\delta_i v_{i+1}(t)}[\mu_i(t) - \mu_{i+1}(t)] - \gamma_i(t)\sigma_{i+1}(t). \tag{3.7}$$

The function $\mu_i^L(t)$ is the drift function and $\gamma_i(t)$ is the volatility function of the forward LIBOR rate $L_i(t)$.

Proof

From equation (3.4) it follows that

$$\frac{dv_{i+1}(t)}{v_{i+1}(t)} = \mu_{i+1}(t)dt + \sigma_{i+1}(t)dz.$$

Now looking at Ito's division rule (Proposition 2.5) we let $\frac{dX}{X} = \frac{dv_i(t)}{v_i(t)}$ and , $\frac{dY}{Y} = \frac{dv_{i+1}(t)}{v_{i+1}(t)}$. Then it follows that

$$\frac{dD}{D} = \frac{d(\frac{X}{Y})}{\frac{X}{Y}} = \frac{d(\frac{v_i(t)}{v_{i+1}(t)})}{\frac{v_i(t)}{v_{i+1}(t)}} = \frac{d(1+\delta_i L_i(t))}{1+\delta_i L_i(t)} = \frac{\delta_i dL_i(t)}{\frac{v_i(t)}{v_{i+1}(t)}} = \mu_D(t)dt + \sigma_D(t)dz.$$

Therefore

$$\frac{\delta_i dL_i(t)}{\frac{v_i(t)}{v_{i+1}(t)}} = [\mu(t) - \mu_{i+1}(t) - \sigma_{i+1}(t)(\sigma_i(t) - \sigma_{i+1}(t))]dt + [\sigma_i(t) - \sigma_{i+1}(t)]dz,$$

which implies that

[2] Refer to section 2.5.3 for the general definition of SDE's and Ito processes.

$$
\begin{aligned}
dL_i(t) &= \frac{v_i(t)}{\delta_i v_{i+1}(t)}\Big[\mu(t)-\mu_{i+1}(t)-\sigma_{i+1}(t)(\sigma_i(t)-\sigma_{i+1}(t))\Big]dt+\frac{v_i(t)}{\delta_i v_{i+1}(t)}\Big[\sigma_i(t)-\sigma_{i+1}(t)\Big]dz \\
&= \frac{v_i(t)}{\delta_i v_{i+1}(t)}\Big[\mu(t)-\mu_{i+1}(t)\Big]-\Big[\frac{v_i(t)}{\delta_i v_{i+1}(t)}(\sigma_i(t)-\sigma_{i+1}(t))\Big]\sigma_{i+1}(t)dt+\frac{v_i(t)}{\delta_i v_{i+1}(t)}\Big[\sigma_i(t)-\sigma_{i+1}(t)\Big]dz \\
&= \frac{v_i(t)}{\delta_i v_{i+1}(t)}\Big[\mu(t)-\mu_{i+1}(t)\Big]-\gamma_i(t)\sigma_{i+1}(t)dt+\gamma_i(t)dz.
\end{aligned}
$$

Therefore

$$
dL_i(t) = \mu_i^L(t)dt + \gamma_i(t)dz, \quad i = 1,2,....N-1, \tag{3.8}
$$

where $\gamma_i(t)$ is defined by equation (3.6) and $\mu_i^L(t)$ by equation (3.7) as required. ■

Now under the no-arbitrage paradigm, we define the forward-neutral probability measure[3] as $P^{T_{i+1}}$, such that the denominated bond price $v_i^{T_{i+1}}(t) = \frac{v_i(t)}{v_{i+1}(t)}$ is a martingale under $P^{T_{i+1}}$. But from equation (3.3) it can be noted that the forward LIBOR rate $L_i(t)$ is a linear function of the denominated bond price, then it can be shown that the forward LIBOR rate is also a martingale under $P^{T_{i+1}}$. The above is summed in the lemma below, with reference to Gumbo([10]).

Lemma 3.1- *For every $i = 1,2,.....N-1$, the LIBOR process L_i is a martingale under the corresponding forward measure $P^{T_{i+1}}$ on the interval $[0,T_i]$.*

Proof

If the forward LIBOR rate is a martingale under $P^{T_{i+1}}$, we need to show that $E_t^{T_{i+1}}[L_i(s)] = L_i(t), \quad t < s \leq T_i$.

From equation(3.3) we have that

$$
L_i(t) = \frac{1}{\delta_i}\bigg(\frac{v_i(t)}{v_{i+1}(t)} - 1\bigg),
$$

$$
\begin{aligned}
E_t^{T_{i+1}}[L_i(s)] &= E_t^{T_{i+1}}\bigg[\frac{1}{\delta_i}\Big(\frac{v_i(s)}{v_{i+1}(s)}-1\Big)\bigg] \\
&= \frac{1}{\delta_i}E_t^{T_{i+1}}\bigg[\frac{v_i(s)}{v_{i+1}(s)}\bigg]-1,
\end{aligned}
$$

but $v_i^{T_{i+1}}(t) = \frac{v_i(t)}{v_{i+1}(t)}$ is a martingale under $P^{T_{i+1}}$, therefore

[3] Refer to subsection 2.6.2, definition 2.13.

$$E_t^{T_{i+1}}[L_i(s)] = \frac{1}{\delta_i}\Big(\frac{v_i(t)}{v_{i+1}(t)} - 1\Big) = L_i(t). \qquad \blacksquare$$

Since the forward LIBOR $L_i(t)$ is a martingale under $P^{T_{i+1}}$, it may be plausible to assume that the process $\{L_i(t)\}$ follows the SDE[4]

$$\frac{dL_i(t)}{L_i(t)} = \gamma_i(t)\, dz^{i+1}, \quad i = 1, 2, ...N-1, \tag{3.9}$$

where $z^{i+1} \equiv \{z^{i+1}(t)\}$ is a standard Brownian motion under $P^{T_{i+1}}$ for some volatility $\gamma_i(t)$.

An important assumption we make for the log-normal LMM is that each of the forward rates $L_i(t)$ are log-normally distributed. Under this assumption the pricing formula for caplets under the LMM will have a structure similar to that of the Black-Scholes model. As a consequence of the above assumption we use Ito's formula (Theorem 2.1) to determine the log-price for the forward LIBOR rate. This is shown below.

Firstly we express equation (3.9) as $dL_i(t) = L_i(t)\gamma_i(t)\, dz^{i+1}$. Now with reference to Ito's formula we obtain the log-price using the transformation $Y(t) = log(L_i(t))$ which implies that $f(x,t) = log(x)$, if and only if , $x = L_i(t)$ therefore

$f_t(x,t) = 0$, $\quad f_x(x,t) = \frac{1}{x} = \frac{1}{L_i(t)}$, $\quad f_{xx}(x,t) = -\frac{1}{x^2} = -\frac{1}{L_i^2(t)}$, $\quad \mu(x,t) = 0$ and $\sigma(x,t) = L_i(t)\gamma_i(t)$.

Thus the SDE for the log-price is then given by

$$\begin{aligned} dlog(L_i(t)) &= [f_t(x,t) + f_x(x,t)\mu(x,t) + \frac{1}{2}f_{xx}(x,t)\sigma^2(x,t)]dt + [f_x(x,t)\sigma(x,t)]dz^{i+1} \\ &= [0 + (\frac{1}{L_i(t)})0 + \frac{1}{2}(-\frac{1}{L_i^2(t)})L_i^2(t)\gamma_i^2(t)]dt + [(\frac{1}{L_i(t)})L_i(t)\gamma_i(t)]dz^{i+1} \\ dlog(L_i(t)) &= -\frac{1}{2}\gamma_i^2(t)dt + \gamma_i(t)dz^{i+1}. \end{aligned}$$

Integrating yields

[4] Refer to Proposition 2.3 and Proposition 2.4 in Section 2.5.3.

$$\begin{aligned} log(L_i(T_i)) - log(L_i(t)) &= \int_t^{T_i} -\frac{1}{2}\gamma_i^2(s)ds + \int_t^{T_i} \gamma_i(s)dz^{i+1} \\ log\left[\frac{L_i(T_i)}{L_i(t)}\right] &= \int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds, \end{aligned} \tag{3.10}$$

where the right hand side of equation (3.10) is a normal random variable with mean $m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds$ and variance $\nu_i^2(t) = \int_t^{T_i} \gamma_i^2(s)ds$[5] . From the above results we can conclude that the LIBOR rate $L_i(T_i)$ is log-normally distributed with mean $m_i(t)$ and variance $\nu_i^2(t)$ under the forward-neutral probability measure $P^{T_{i+1}}$. The mean and the variance stated above can be determined as follows.

Given that the forward LIBOR rates are $\mathcal{F}_t$-measurable, then the mean under $P^{T_{i+1}}$ is given by

$$\begin{aligned} E_t^{T_{i+1}}\left[log\Big(\frac{L_i(T_i)}{L_i(t)}\Big)\right] &= E_t^{T_{i+1}}\left[\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds\right] \\ &= E_t^{T_{i+1}}\left[\int_t^{T_i} \gamma_i(s)dz^{i+1}\right] - E_t^{T_{i+1}}\left[\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds\right] \\ &= 0 - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds \\ &= -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds, \end{aligned}$$

where we used the property (3) of the Ito integral[6] and the assumption that the volatility function $\gamma_i(t)$ is a deterministic function of time.

Thus the mean is given by

$$m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds.$$

For the variance we proceed as follows. Using the definition of variance, we have that

$$Var\left[log(\frac{L_i(T_i)}{L_i(t)})\right] = E^{T_{i+1}}\left[\Big(\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds\Big) - E^{T_{i+1}}\Big(\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds\Big)\right]^2.$$

[5]Note that $m_i(t) \equiv m(t, T_i)$ and $\nu_i(t) \equiv \nu(t, T_i)$.

[6]Refer to the properties of the Ito integral outlined in section 2.5.2.

But the mean is given by $m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds$, thus

$$\begin{aligned}
Var\Big[log\Big(\frac{L_i(T_i)}{L_i(t)}\Big)\Big] &= E^{T_{i+1}}\Big[(\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds) - (-\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds)\Big]^2 \\
&= E^{T_{i+1}}\Big[\int_t^{T_i} \gamma_i(s)dz^{i+1} - \frac{1}{2}\int_t^{T_i} \gamma_i^2(s)ds + \frac{1}{2}\int_t^{T_i} \gamma_i^2(s)ds)\Big]^2 \\
&= E^{T_{i+1}}\Big[\int_t^{T_i} \gamma_i(s)dz^{i+1}\Big]^2 \\
&= \int_t^{T_i} E^{T_{i+1}}\Big[\gamma_i^2(s)\Big]ds, \text{ by Ito isometry (see section 2.5.2).}
\end{aligned}$$

Thus

$$Var\Big[log\Big(\frac{L_i(T_i)}{L_i(t)}\Big)\Big] = \int_t^{T_i} \gamma_i^2(s)ds, \text{ since the } \gamma_i(t)\text{'s are deterministic.}$$

Therefore the variance of the log ratio of the forward LIBOR rates, $Var\Big[log\Big(\frac{L_i(T_i)}{L_i(t)}\Big)\Big]$, denoted by $\nu_i^2(t)$ for simplicity is given by

$$\nu_i^2(t) = \int_t^{T_i} \gamma_i^2(s)ds.$$

As a result equation (3.10) can be written as

$$L_i(T_i) = L_i(t)e^{\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds}, \tag{3.11}$$

which is an explicit solution to equation (3.9). Also from the above equation we can conclude that $L_i(t)$ is simply a geometric Brownian motion with drift function $m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds$ and volatility function $\nu_i(t) = \sqrt{\int_t^{T_i} \gamma_i^2(s)ds}$.

In conclusion, the dynamics of the forward LIBOR $L_i(t)$ outlined above imply the existence of the LMM, following some technical assumptions. This is summarized in the proposition below.

Proposition 3.1 *Suppose the process $\{L_i(t)\}$ is a solution to the SDE of forward LIBOR rates $L_i(t),\ i = 1, ..., N-1$, defined by*

$$dL_i(t) = \mu_i^L(t)dt + \gamma_i(t)dz, \quad i = 1, .., N-1,$$

where $\mu_i^L(t)$ is the drift function , $\gamma_i(t)$ is the volatility function and $z = \{z(t)\}$ is a standard Brownian motion under empirical probability P. Then the log-normal LIBOR Market Model exists if and only if,

1. *The forward LIBOR rate $L_i(t)$ is given by equation (3.3) and the initial term structure $L_i(0)$ is known.*

2. *$L_i(t)$ is a martingale under forward probability measure $P^{T_{i+1}}$ with SDE equivalent to equation (3.9), for some volatility $\gamma_i(t)$, where $z^{i+1} = \{z^{i+1}(t)\}$ is a standard Brownian motion under $P^{T_{i+1}}$.*

3. *The volatility function $\gamma_i(t),\ t \leq T_i$ is a deterministic function of time $t \leq T_i$, for each settlement date i.e. the forward LIBOR rates $L_i(t),\ i = 1, .., N-1$, are log-normally distributed with mean, $m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds$ and variance, $\nu_i^2(t) = \int_t^{T_i} \gamma_i^2(s)ds$.*

Having stated the existence of the LMM in the above proposition, we are now in a position to use it to price caplets in the market. Finally we summarise a list of inputs required for the LMM and they are shown below following Pietersz ([24]).

Inputs:

- A set of default-free discount bond prices $v_i(t)$ with corresponding maturity dates $T_i,\ i = 1, 2, ..., N$.
- Time zero ($t = 0$) forward LIBOR rates $L_1(0), L_2, ..., L_{N-1}(0)$.
- Instantaneous volatilities of the forward LIBOR rates $\gamma_i(.),\quad i = 1, ..., N-1$.

3.2 Terminal Measure dynamics for the forward LIBOR rates.

In the previous section we showed that the forward LIBOR process L_i $i = 1, .., N-1$ is a martingale under the corresponding forward measure $P^{T_{i+1}}$ with respect to each forward time T_{i+1}. We also stated without proof the existence of the LMM. This section shows how to price complex derivatives that use more than just one forward LIBOR rate and generate payoffs at more than one time interval , for example caplets with relevance to this thesis. This entails that we find the dynamics of all our forward rates under a single measure, known as the terminal measure, denoted by P^{T_N} . The idea behind the LMM is to model the forward LIBOR processes under the terminal measure, where the choice of numeraire is the bond with the longest maturity, i.e. $v(t, T_N) \equiv v_N(t)$ and $z^N(t) \equiv z^N$ is the terminal measure standard Brownian motion. Under the terminal measure, the terminal LIBOR $L_{N-1}(t)$ is a martingale (Kwok[18]). The construction of the SDE for forward LIBOR rates under terminal measure is outlined below.[7] Such a construction leads to restrictions on the drift term of the forward LIBOR rates. We show how to apply Girsanov's theorem to find the successive standard Brownian motion under respective measures $P^{T_{N-1}}, P^{T_{N-2}}, ..., P^{T_1}$.

Now using equation (3.2), the Radon-Nikodym derivative[8] that effects the change of measure from P^{T_i} to $P^{T_{i+1}}$, for the forward LIBOR rate is given by

$$\frac{dP^{T_i}}{dP^{T_{i+1}}} = \frac{v_i(t)/v_i(0)}{v_{i+1}(t)/v_{i+1}(0)} = \frac{v_i(t)}{v_{i+1}(t)} \frac{v_{i+1}(0)}{v_i(0)},$$

which implies that

$$\eta_i(t) = \frac{dP^{T_i}}{dP^{T_{i+1}}} = c\left(\frac{v_i(t)}{v_{i+1}(t)}\right) = c(1 + \delta_i L_i(t)), \tag{3.12}$$

where $c = \frac{v_{i+1}(0)}{v_i(0)}$ is a normalizing constant.

Now using Girsanov's theorem[9] it follows that

$$dz^i = dz^{i+1} + \beta_i(t)dt, \tag{3.13}$$

where $\beta_i(t)$ satisfies

$$\eta_i(t) \quad = \quad exp\Big(\int_0^t -\beta_i(s)dz^{i+1} - \frac{1}{2}\int_0^t \beta^2(s)dt\Big). \tag{3.14}$$

From equation (3.12) and (3.14) it follows that

[7] The construction below is based on the work done by Kwok ([18]) and Blackham ([3]).
[8] See definition 2.14 in subsection 2.7.1.
[9] See Theorem 2.6, section 2.7.

$$\eta_i(t) = c(1+\delta_i L_i(t)) = exp\Big(\int_0^t -\beta_i(s)dz^{i+1} - \frac{1}{2}\int_0^t \beta^2(s)dt\Big),$$

and in differential form

$$d\eta_i(t) = c\delta_i dL_i(t) = -\eta_i(t)\beta_i(t)dz^{i+1}.$$

Substituting equation (3.9) and (3.12) into the above equation implies that

$$c\delta_i L_i(t)\gamma_i(t)dz^{i+1} = -c(1+\delta_i L_i(t))\beta_i(t)dz^{i+1},$$

and therefore

$$\beta_i(t) = -\frac{\delta_i L_i(t)\gamma_i(t)}{1+\delta_i L_i(t)}. \tag{3.15}$$

Hence we can conclude from equation (3.13) and equation (3.15) that

$$dz^i = dz^{i+1} - \frac{\delta_i L_i(t)\gamma_i(t)}{1+\delta_i L_i(t)}dt. \tag{3.16}$$

Recall that the forward LIBOR rate $L_{N-1}(t)$ is a martingale under terminal measure P^{T_N}. The example below shows that $L_{N-2}(t)$ is a martingale under $P^{T_{N-1}}$ but not a martingale under P^{T_N}.

Using equation (3.16) it follows that

$$\begin{aligned}
dL_{N-2}(t) &= L_{N-2}(t)\gamma_{N-2}(t)dz^{N-1}, \quad \text{which is a martingale.} \\
&= L_{N-2}(t)\gamma_{N-2}(t)\left[dz^N - \frac{\delta_{N-1}L_{N-1}(t)\gamma_{N-1}(t)}{1+\delta_{N-1}L_{N-1}(t)}dt\right] \\
dL_{N-2}(t) &= L_{N-2}(t)\left[-\frac{\delta_{N-1}L_{N-1}(t)\gamma_{N-1}(t)\gamma_{N-2}(t)}{1+\delta_{N-1}L_{N-1}(t)}dt\right] + L_{N-2}(t)\gamma_{N-2}(t)dz^N.
\end{aligned}$$

From the results shown above, Jamshidan ([12]) suggests that we can deductively obtain the forward rates $L_{N-2}, ..., L_1$ and if the solution $\{L_i(t)\}$ exists then the i^{th}-component of the forward LIBOR rate follows the SDE

$$dL_i(t) = L_i(t)\mu_i^{term}(t)dt + L_i(t)\gamma_i(t)dz^N, \quad i = 1, .., N-1, \tag{3.17}$$

where

$$\mu_i^{term}(t) = \begin{cases} -\sum_{j=i+1}^{N-1} \frac{\delta_j L_j(t)\gamma_j'(t)\gamma_i(t)}{1+\delta_j L_j(t)} & \text{for } i < N-1 \\ 0 & \text{for } i = N-1 \end{cases},$$

and $z^N \equiv z^N(t)$ is a standard Brownian motion under the terminal measure P^{T_N}. Hence given the P^{T_N}-processes of these $N-1$ forward LIBOR rates, all numeraire denominated bond prices can be determined (De Jong et.al, [7]).

Now that we have outlined the dynamics for the LMM for simple and complex interest rate derivatives, we are now in a position to implement the LMM to pricing of caplets which will be discussed in the following section.

3.3 Pricing of Caplets.

In this section we discuss the pricing of caplets using the LMM. We show that arbitrage-free LMM pricing formula for caplets has a similar structure to that of the Black-Scholes pricing formula for caplets. Assuming that the LMM exists (as developed in the earlier section), we first derive the Black-Scholes formula for pricing caplets and then construct the LMM price of the same caplet. Before proceeding we state the mathematical definition of a caplet with reference to notation used by Kijima ([14]).

Definition 3.2: Caplet

Let $L_i(t) \equiv L(t, T_i, T_{i+1})$ denote the LIBOR rate, that covers the period $[T_i, T_{i+1}]$, where T_i is the time epoch that the i^{th} payment is made. Then an interest derivative whose payoff at time T_{i+1} is given by

$$\delta_i\{L_i(T_i) - K\}_+, \quad \delta_i = T_{i+1} - T_i, \tag{3.18}$$

where

$$\{x\}_+ = max\{x, 0\} = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{3.19}$$

is called a *caplet,* where K is called the cap rate.

The forward LIBOR rate $L_i(t)$ is the reference floating rate and the cap rate (K) is the fixed rate for this forward rate agreement. From the definition above a caplet can be viewed as a call option written on the forward LIBOR rate with cap rate K. A strip or a collection of caplets is called a cap i.e. a cap is a sum of caplets.

Let $Cpl_i(T_{i+1})$ denote the payoff of the i^{th} caplet received at time T_{i+1} , then $Cpl_i(T_{i+1}) = \delta_i\{L_i(T_i) - K\}_+$. The caplet payoff is known at time T_i but received at time T_{i+1}. Thus for forward LIBOR rates $L_i(.)$, $i = 1, 2, ..., N-1$ we have the following vector of corresponding caplet payoffs for $i = 1, 2, ..., N-1$ shown below.

$$
\begin{aligned}
Cpl_1(T_2) &= \delta_1\{L_1(T_1) - K\}_+ \\
Cpl_2(T_3) &= \delta_2\{L_2(T_2) - K\}_+ \\
&. \quad . \quad . \\
&. \quad . \quad . \\
&. \quad . \quad . \\
Cpl_{N-1}(T_N) &= \delta_{N-1}\{L_{N-1}(T_{N-1}) - K\}_+,
\end{aligned}
$$

where $Cpl_{N-1}(T_N)$ denotes the payoff at maturity T_N for a caplet contracted at time T_{N-1}.

Now as discussed in section 2.6.2, we can determine the arbitrage-free price of a caplet under the forward-neutral probability measure. Let $Cpl_i(t)$ denote the time t price price of the i^{th} caplet. Then with reference to equation (2.21) we can determine the price of the caplet. Let $h(S(T)) = h(Cpl_i(T_{i+1})) = \Big[\delta_i\{L_i(T_i) - K\}_+\Big]$ denote the payoff function of the caplet at time T_{i+1}, $v(t, T_{i+1}) = v_{i+1}(t)$ denote the time t price of a default-free discount bond maturing at time T_{i+1} and $E_t^{T_{i+1}}$ denote the conditional expectation under the forward-neutral probability measure $P^{T_{i+1}}$, given filtration $\mathcal{F}_t$. Then the time t price of the i^{th} caplet is given by

$$Cpl_i(t) = v(t, T_{i+1}) E_t^{T_{i+1}} \Big[h(Cpl_i(T_{i+1}))\Big].$$

Therefore

$$Cpl_i(t) = \delta_i v_{i+1}(t) E_t^{T_{i+1}} \Big[\{L_i(T_i) - K\}_+\Big], \quad 0 \le t \le T_i. \tag{3.20}$$

From equation (3.20) we can subsequently derive the Black-Scholes formula for a caplet. In the next subsection we state the Black-Scholes pricing formula for caplets.

3.3.1 Pricing caplets using Black- Scholes formula.

Using the general Black-Scholes formula discussed in Chapter 2, section 2.8 , we can construct the Black-Scholes formula for caplets. The next proposition states the Black-Scholes formula for pricing a single (i^{th}) caplet.

Proposition 3.2 Black Scholes formula for caplets.

Let $Cpl_i^{BLACK}(t)$ *denote the time* t *price of a caplet. Then the Black-Scholes price of a caplet written on the forward LIBOR rate* $L_i(t)$ *with maturity time* T_i *and cap rate* K *is given by*

$$Cpl_i^{BLACK}(t) = \delta_i v_{i+1}(t)\Big[L_i(t)\Phi(d_i) - K\Phi\big(d_i - \sigma_{BLACK}\sqrt{T_i - t}\big)\Big], \tag{3.21}$$

where

$$d_i = \frac{log\Big(\frac{L_i(t)}{K}\Big) + \frac{1}{2}\sigma_{BLACK}^2(T_i - t)}{\sigma_{BLACK}\sqrt{T_i - t}},$$

σ_{BLACK} *denotes the volatility of the forward LIBOR rate* $L_i(t)$ *and* $\Phi(d_i)$ *is the distribution function of the standard normal distribution.*

Remark: The above stated formula can be easily verified using the Black-Scholes formula for call options given in Theorem 2.7 in section 2.8. This is achieved by letting $c(S,t) = Cpl_i^{BLACK}(t)$, $S(t) = \delta_i L_i(t)$, $\sigma\sqrt{T-t} = \sigma_{BLACK}\sqrt{T_i - t}$ and $B(t) = v_{i+1}(t)$.

3.3.2 Pricing caplets using the LIBOR Market Model (LMM).

Having outlined the Black-Scholes pricing formula for caplets in the previous section, this section illustrates how to price caplets under the LIBOR Market Model (LMM). We show that the resulting pricing formula coincides with that of the Black-Scholes model, in structure. Recall that the payoff of the i^{th} caplet received at time T_{i+1} is given by $Cpl_i(T_{i+1}) = \delta_i\{L_i(T_i) - K\}_+$. Under the no-arbitrage paradigm the time t price of the i^{th} caplet, under terminal measure P^{T_N} is given by $Cpl_i(t) = \delta_i v_{i+1}(t) E_t^{T_{i+1}}\Big[\{L_i(T_i) - K\}_+\Big]$. From this payoff function we are able to construct the LMM price of the caplet using the properties of the LMM. The fair price of a caplet derived using the LMM is given in the proposition below.

Proposition 3.3 *Consider the payoff function of the i^{th} caplet accrued for the interval $[T_i, T_{i+1}]$ given by $\delta_i\{L_i(T_i) - K\}_+$. Then using the LIBOR Market Model (LMM), the time t price of the caplet denoted by $Cpl_i^{LMM}(t)$, written on the forward LIBOR rate $L_i(t)$ with maturity time T_i and cap rate K is given by*

$$Cpl_i^{LMM}(t) = \delta_i v_{i+1}(t)\Big[L_i(t)\Phi(d_i) - K\Phi(d_i - \nu_i(t))\Big], \tag{3.22}$$

where

$$d_i = \frac{log\Big(\frac{L_i(t)}{K}\Big) + \frac{1}{2}\nu_i^2(t)}{\nu_i(t)},$$

and $\nu_i(t) = \sqrt{\int_t^{T_i} \gamma_i^2(s)ds}$ denotes the volatility function of the forward LIBOR rate $L_i(t)$.

Proof

The time t price of the i^{th}caplet under terminal measure is given by

$$\delta_i v_{i+1}(t) E_t^{T_{i+1}}\Big[\{L_i(T_i) - K\}_+\Big].$$

Similarly the LMM price the same caplet can be written as

$$Cpl_i^{LMM}(t) = \delta_i v_{i+1}(t) E_t^{T_{i+1}}\Big[\{L_i(T_i) - K\}_+\Big].$$

Now under the LMM, the forward LIBOR rate $L_i(T_i)$ can be explicitly solved as $L_i(T_i) = L_i(t)e^{\int_t^{T_i} \gamma_i(s)dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds}$ as shown in equation (3.11) which implies that $L_i(T_i)$ is log-normally distributed with mean $m_i(t) = -\int_t^{T_i} \frac{1}{2}\gamma_i^2(s)ds$ and variance $\nu_i^2(t) = \int_t^{T_i} \gamma_i^2(s)ds$. In particular if we let $\nu_i \equiv \nu_i(t) = \sqrt{\int_t^{T_i} \gamma_i^2(s)ds}$ for simplicity, then $log\Big[\frac{L_i(T_i)}{L_i(t)}\Big] \sim N(-\frac{1}{2}\nu_i^2, \nu_i^2)$. Also under the LMM we assume that the volatility function $\gamma_i(t)$ is deterministic function of time, therefore we can conclude that $\int_t^{T_i} \gamma_i(s)dz^{i+1}$ is normally distributed with mean 0 and variance $\nu_i^2(t)$, i.e.

$$\int_t^{T_i} \gamma_i(s)dz^{i+1} \sim N\Big(0, \int_t^{T_i} \gamma_i^2(s)ds\Big),$$

which implies that

$$\nu_i N(0,1) \sim N\Big(0, \int_t^{T_i} \gamma_i^2(s)ds\Big).$$

Substituting the above results it follows that

$$\begin{aligned} Cpl_i^{LMM}(t) &= \delta_i v_{i+1}(t) E_t^{T_{i+1}} \Big[\{ L_i(t) e^{\int_t^{T_i} \gamma_i(s) dz^{i+1} - \int_t^{T_i} \frac{1}{2}\gamma_i^2(s) ds} - K \}_+ \Big] \\ &= \delta_i v_{i+1}(t) E_t^{T_{i+1}} \Big[\{ L_i(t) e^{-\int_t^{T_i} \frac{1}{2}\gamma_i^2(s) ds + \nu_i Y} - K \}_+ \Big], \end{aligned}$$

where $Y \sim N(0,1)$.

Therefore

$$\begin{aligned} Cpl_i^{LMM}(t) &= \delta_i v_{i+1}(t) E_t^{T_{i+1}} \Big[\{ L_i(t) e^{-\frac{1}{2}\nu_i^2 + \nu_i Y} - K \}_+ \Big] \\ &= \delta_i v_{i+1}(t) \Big[\int_{-\infty}^{\infty} \{ L_i(t) e^{-\frac{1}{2}\nu_i^2 + \nu_i y} - K \}_+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \Big]. \end{aligned}$$

At maturity of the i^{th} caplet, a payoff of $L_i(T_i) - K$ is received if and only if $L_i(T_i) \geq K$ which implies that

$$\begin{aligned} L_i(T_i) &\geq K \\ L_i(t) e^{-\frac{1}{2}\nu_i^2 + \nu_i y} &\geq K \\ e^{-\frac{1}{2}\nu_i^2 + \nu_i y} &\geq \frac{K}{L_i(t)} \\ e^{\nu_i y} &\geq \frac{K}{L_i(t)} e^{\frac{1}{2}\nu_i^2} \\ \nu_i y &\geq log\bigg(\frac{K}{L_i(t)}\bigg) + \frac{1}{2}\nu_i^2 \\ y &\geq \frac{log\Big(\frac{K}{L_i(t)}\Big) + \frac{1}{2}\nu_i^2}{\nu_i}, \end{aligned}$$

and if we let $\omega_1 = \frac{log\left(\frac{K}{L_i(t)}\right) + \frac{1}{2}\nu_i^2}{\nu_i}$, then

$$
\begin{aligned}
Cpl_i^{LMM}(t) &= \delta_i v_{i+1}(t)\Big[\int_{\omega_1}^{\infty}\Big(L_i(t)e^{-\frac{1}{2}\nu_i^2+\nu_i y}-K\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\Big] \\
&= \delta_i v_{i+1}(t)\Big[L_i(t)\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\nu_i^2+\nu_i y-\frac{y^2}{2}}dy - K\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\Big] \\
&= \delta_i v_{i+1}(t)\Big[L_i(t)\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-2\nu_i y+\nu_i^2)}dy - K(1-\Phi(\omega_1))\Big] \\
&= \delta_i v_{i+1}(t)\Big[L_i(t)\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\nu_i)^2}dy - K(1-\Phi(\omega_1))\Big], \quad \text{by completing the square.}
\end{aligned}
$$

Let $x = y - \nu_i$ and $\omega_2 = \omega_1 - \nu_i$, then it follows that

$$
\begin{aligned}
Cpl_i^{LMM}(t) &= \delta_i v_{i+1}(t)\Big[L_i(t)\int_{\omega_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx - K(1-\Phi(\omega_1))\Big] \\
&= \delta_i v_{i+1}(t)\Big[L_i(t)(1-\Phi(\omega_2)) - K(1-\Phi(\omega_1))\Big] \\
&= \delta_i v_{i+1}(t)\Big[L_i(t)(-\omega_2) - K\Phi(-\omega_1)\Big], \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(-\omega_2) &= \Phi\left(\frac{log\Big(\frac{L_i(t)}{K}\Big)+\frac{1}{2}\nu_i^2}{\nu_i}\right) \\
\Phi(-\omega_1) &= \Phi\left(\frac{log\Big(\frac{L_i(t)}{K}\Big)+\frac{1}{2}\nu_i^2}{\nu_i}-\nu_i\right).
\end{aligned}
$$

Substituting back into equation (3.23) , it follows that the fair price of a caplet within the LMM is given by

$$
Cpl_i^{LMM}(t) = \delta_i v_{i+1}(t)\Big[L_i(t)\Phi(d_i) - K\Phi(d_i-\nu_i)\Big],
$$

where

$$
d_i = \frac{log\Big(\frac{L_i(t)}{K}\Big)+\frac{1}{2}\nu_i^2}{\nu_i},
$$

and $\nu_i \equiv \nu_i(t) = \sqrt{\int_t^{T_i}\gamma_i^2(s)ds}$. $\blacksquare$

Remarks: From the above we can easily observe that the LMM price of a caplet has a similar structure to that of the Black-Scholes formula when $\sigma_{BLACK}\sqrt{T_i - t} = \nu_i(t)$, that is $\sigma_{BLACK} = \frac{1}{\sqrt{T_i - t}}\sqrt{\int_t^{T_i} \gamma_i^2(s)ds}$.

In conclusion one can effectively use the LMM to price caplets and this would lead to caplet prices which are more or less the same as those computed using the Black-Scholes model. In the next chapter we develop the Swap Market Model (SMM) which can be used to calculate arbitrage-free prices of swaptions. We shall proceed in a similar fashion to what was done in this chapter. We will show that the SMM pricing formula for swaptions will have the same structure as that of the Black-Scholes pricing formula for swaptions.

Chapter 4

Swap Market Model

In Chapter 4 we discuss the construction of the Swap Market Model (SMM) proposed by Jamshidan ([12]). The SMM can be used to price interest rate derivatives such as swaps and swaptions. De Jong et.al ([7]) suggests that the SMM can be constructed in a way that is quite similar to the construction of the LMM. The idea behind the SMM is to construct an arbitrage-free interest rate model that implies a pricing formula for swaptions that has the same structure as the Black-Scholes pricing formula for swaptions. The underlying asset used in the construction of the SMM is the forward swap rate. We show that the dynamics of the forward swap rate are a martingale under some forward swap probability measure. We discuss the dynamics of the forward swap rate under the co-terminal swap measure, but the algebra is more complicated than the terminal measure dynamics of the forward LIBOR rate discussed in Chapter 3. As a result we refer to Jamshidan ([12]) for a detailed discussion. Before we develop the SMM we shade light on interest rate swaps which will lead to the definition of the forward swap rate. We shall also define a swaption via the definition of an interest rate swap and the forward swap rate.

An *interest rate swap* (or just a swap) is a financial contract between two parties who agree to exchange cash flows equivalent to a reference interest rate, based on some notional principal[1] at a future date T. An example of an interest rate swap is the "plain vanilla" swap in which one party pays a fixed rate (floating rate receiver) and the other party pays a floating rate (fixed rate receiver). The party that pays the fixed rate is called the *payer* and the other party paying the floating rate is called the *receiver* of the swap (Leung Lai and Xing, [20]). The interest rate that makes the initial value of the contract equal to zero is called the *swap rate*. The floating rate is usually the LIBOR rate.

[1] Here we shall assume that the notional principal is unity i.e. 1.

4.1 Dynamics of the forward swap process: Swap Market Model.

Consider the tenor structure $0 = T_0 < T_1 < ... < T_i < T_{i+1} < ... < T_N$, with tenor $\delta_i = T_{i+1} - T_i,\ i = 1, 2, ..., N-1$, where $\{T_0, T_1, ..., T_{N-1}\}$ are the reset dates at which relevant forward LIBOR rates, $\{L_0, L_1, ..., L_{N-1}\}$ are determined and their payment dates are $\{T_1, T_2, ..., T_N\}$. Let $L_i(t)$ define the time $t \leq T_i$ value of the forward LIBOR rate contracted for period $[T_i, T_{i+1}]$, given in equation (3.3) . Now consider the preassigned payment dates of an interest rate swap $T_1, T_2, ..., T_N$, where the floating rate interest payments are exchanged for a fixed rate interest payments and T_N is the maturity date of the swap. At time T_i, the fixed rate receiver, receives a fixed interest payment $\delta_i K$, where K is the fixed interest rate[2]. At time T_{i+1}, the floating rate receiver receives the floating interest payment $\delta_i L_i(T_i)$, where $L_i(T_i) \equiv L(T_i, T_{i+1})$ denotes the forward LIBOR rate contracted at time T_i for the period $[T_i, T_{i+1}]$. As mentioned in Chapter 3, one cannot trade n units of the forward LIBOR rate in the market, hence default-free discount bonds can be used as underlying assets for the forward LIBOR rate. The time t price of the discount bond maturing at time T_i is given by $v(t, T_i) \equiv v_i(t),\ i = 1, ..., N$. At maturity, the bond pays an amount equal to unity i.e. $v(T_i, T_i) = v_i(T_i) = 1$. We assume that the N default-free discount bonds follow the Ito processes under empirical probability P . Now to ensure that there is no arbitrage opportunities in the market, the fixed rate K has to be chosen at time T_i such that the present (time t) value of the swap is equal to zero. Such a choice of K is known as the swap rate. We determine the present value of the fixed and floating side under the forward-neutral probability measure $P^{T_{i+1}}$.

Consider the interest paid on the fixed side given by $\delta_i K$. Now assuming there are no arbitrage opportunities in the market, the present value (time $t \leq T_i$) for the fixed side denoted by $V^{FIX}(t, T_i) \equiv V_i^{FIX}(t)$, is given by[3]

$$
\begin{aligned}
V_i^{FIX}(t) &= v_{i+1}(t) E_t^{T_{i+1}}\left[\frac{\delta_i K}{v_{i+1}(T_{i+1})}\right] \\
&= v_{i+1}(t) E_t^{T_{i+1}}[\delta_i K], \text{ since } v_{i+1}(T_{i+1}) = 1.
\end{aligned}
$$

Therefore the present value for the fixed side is given by

$$
V_i^{FIX}(t) = K \delta_i v_{i+1}(t). \tag{4.1}
$$

[2]The variable K is not the same as the variable K (strike price) discussed in the earlier chapters.

[3]This is in the same light of pricing contingent claims under the forward neutral probability measure discussed in Chapter 2, subsection 2.6.2, where the payoff $h(S(T)) = \delta_i K$.

Similarly the present value (time $t \leq T_{i+1}$) for the floating side under $P^{T_{i+1}}$, denoted by $V^{FLOAT}(t, T_{i+1}) \equiv V^{FL}_{i+1}(t)$ is given by

$$
\begin{aligned}
V^{FL}_{i+1}(t) &= v_{i+1}(t) E^{T_{i+1}}_t \Big[\frac{\delta_i L_i(T_i)}{v_{i+1}(T_{i+1})} \Big] \\
&= \delta_i v_{i+1}(t) E^{T_{i+1}}_t \Big[L_i(T_i) \Big], \text{ since } v_{i+1}(T_{i+1}) = 1.
\end{aligned}
$$

Therefore the present value for the floating side is given by

$$
V^{FL}_{i+1}(t) = \delta_i v_{i+1}(t) L_i(t),
$$

since $\{L_i(t)\}$ is a martingale under $P^{T_{i+1}}$.

It then follows from equation (3.3), that

$$
\begin{aligned}
V^{FL}_{i+1}(t) &= \delta_i v_{i+1}(t) \left[\frac{1}{\delta_i} \left(\frac{v_i(t)}{v_{i+1}(t)} - 1 \right) \right] \\
&= \delta_i v_{i+1}(t) \left(\frac{v_i(t) - v_{i+1}(t)}{\delta_i v_{i+1}(t)} \right).
\end{aligned}
$$

Therefore the time t value of the floating side is given by

$$
V^{FL}_{i+1}(t) = v_i(t) - v_{i+1}(t). \tag{4.2}
$$

Definition 4.1: Value of a payer swap

Let $V^{PS}(t, T_n, T_N) \equiv V^{PS}_{n,N}(t)$ denote the time t value of a payers swap, for $t \leq T_n$, that starts at time T_n, $n = 1, 2, ..., N-1$, and ends at time T_N . Then the present value of the payer swap is given by

$$
V^{PS}_{n,N}(t) = \sum_{i=n}^{N-1} V^{FL}_{i+1}(t) - \sum_{i=n}^{N-1} V^{FIX}_i(t), \tag{4.3}
$$

where the cash flows are exchanged at dates $T_{n+1}, ..., T_N$ for the swap tenor period $T_N - T_n$.

Remark: The holder of the payer swap pays a fixed a fixed rate and receives a floating rate.

It follows from equations (4.1) and (4.2) that

$$\begin{aligned} V^{PS}_{n,N}(t) &= \sum_{i=n}^{N-1}\Big(v_i(t)-v_{i+1}(t)\Big)-\sum_{i=n}^{N-1}K\delta_i v_{i+1}(t) \\ V^{PS}_{n,N}(t) &= \Big(v_n(t)-v_N(t)\Big)-K\sum_{i=n}^{N-1}\delta_i v_{i+1}(t). \end{aligned} \tag{4.4}$$

Definition 4.2: Value of a receiver swap

Let $V^{RS}(t,T_n,T_N)\equiv V^{RS}_{n,N}(t)$ denote the time t value of a receiver swap that starts at time $T_n,\ n=1,2,...,N-1$, and ends at time T_N . Then the present value of the receiver swap is given by

$$V^{RS}_{n,N}(t)=\sum_{i=n}^{N-1}V_i^{FIX}(t)-\sum_{i=n}^{N-1}V_i^{FL}(t), \tag{4.5}$$

where the cash flows are exchanged at dates $T_{n+1},...,T_N$ for the swap tenor period T_N-T_n.

Remark: The holder of a receiver swap pays a floating rate and receives a fixed rate.

Similarly, it follows from equations (4.1) and (4.2) that

$$\begin{aligned} V^{RS}_{n,N}(t) &= \sum_{i=n}^{N-1}K\delta_i v_{i+1}(t)-\sum_{i=n}^{N-1}\Big(v_i(t)-v_{i+1}(t)\Big) \\ V^{RS}_{n,N}(t) &= K\sum_{i=n}^{N-1}\delta_i v_{i+1}(t)-\Big(v_n(t)-v_N(t)\Big). \end{aligned} \tag{4.6}$$

Definition 4.3: Forward Swap rate

Let $S(t,T_n,T_N)\equiv S_{n,N}(t)$ denote the time t *forward swap rate* or the *par swap* for the payer or receiver swap. Then the forward swap rate is the fixed rate K, such that the present value of the swap (payer or receiver) is equal to zero, that is $V^{PS}_{n,N}(t)=V^{RS}_{n,N}(t)=0$, is given by

$$S_{n,N}(t)=\frac{v_n(t)-v_N(t)}{\sum_{i=n}^{N-1}\delta_i v_{i+1}(t)},$$

with initial term structure $S_{n,N}(0)$. The sum, $\sum_{i=n}^{N-1}\delta_i v_{i+1}(t)\equiv\sum_{i=n+1}^{N}\delta_{i-1}v_i(t)$ is commonly called the *present value of a basis point* (PVBP) in the financial market (Kwok, [18]).

Choudhry ([6]) defines the PVBP as the change in the bond price for one basis point change in the bond's yield[4]. To simply things we denote the PVBP as $P_{n+1,N}(t)$ and hence the forward swap rate is given by

$$S_{n,N}(t) = \frac{v_n(t) - v_N(t)}{P_{n+1,N}(t)}. \tag{4.7}$$

Remarks:

1. For a one period swap starting at time T_i and ends at time time T_{i+1}, the one period forward swap rate, $S(t, T_i, T_{i+1}) \equiv S_{i,i+1}(t)$ is simply the forward LIBOR rate $L_i(t)$ given by equation (3.3), i.e.

$$\begin{aligned}
S_{i,i+1}(t) &= \frac{v_i(t) - v_{i+1}(t)}{P_{i+1,i+1}(t)} \\
&= \frac{v_i(t) - v_{i+1}(t)}{\sum_{k=i+1}^{i+1} \delta_{k-1} v_k(t)} \\
&= \frac{v_i(t) - v_{i+1}(t)}{\delta_i v_{i+1}(t)} \\
&= \frac{1}{\delta_i}\left(\frac{v_i(t)}{v_{i+1}(t)} - 1\right) \\
&= L_i(t).
\end{aligned}$$

2. From equations (4.4) and (4.7), it follows that the present value of the payers swap, for the different fixed rates K, is given by

$$\begin{aligned}
V_{n,N}^{PS}(t) &= \Big(v_n(t) - v_N(t)\Big) - K \sum_{i=n}^{N-1} \delta_i v_{i+1}(t) \\
&= \Big(v_n(t) - v_N(t)\Big) - K P_{n+1,N}(t) \\
&= S_{n,N}(t) P_{n+1,N}(t) - K P_{n+1,N}(t) \\
V_{n,N}^{PS}(t) &= P_{n+1,N}(t)\Big(S_{n,N}(t) - K\Big). \qquad (4.8)
\end{aligned}$$

3. In a similar fashion to the above, from equations (4.6) and (4.7) it follows that the present value of the receivers swap for the different fixed rates K, is given by

$$V_{n,N}^{RS}(t) = P_{n+1,N}(t)\Big(K - S_{n,N}(t)\Big). \tag{4.9}$$

The forward swap rate $S_{n,N}(t)$ is the underlying asset in the construction of the log-normal

[4]Here we note that one basis point change in the bond's yield implies a 0.01% change in the bond's yield.

SMM, which will be used to determine the non-arbitrage price of swaptions. We note that the forward swap rate is a function of the time t default-free discount bonds with maturities $T_i,\ i = 1, 2, ..., N$. We shall assume that the discount bonds follow Ito processes under empirical probability P, with stochastic differential equation (SDE)[5]

$$\frac{dv_i(t)}{v_i(t)} = \mu_i(t)dt + \sigma_i(t)dz, \tag{4.10}$$

where $\mu_i(t)$ denotes the drift function and $\sigma_i(t)$ the volatility function. We also assume under empirical probability that the set of forward swap rates $S_{n,N}(t)$, starting at times $T_n, n = 1, ..., N-1$ and with same maturity date T_N follows the SDE

$$dS_{n,N}(t) = \mu_{n,N}(t)dt + \gamma_{n,N}(t)dz, \quad n = 1, 2, ...N-1, \tag{4.11}$$

where $\mu_{n,N}(t)$ and $\gamma_{n,N}(t)$ denotes the drift and volatility function of the forward swap rate, respectively.

Now under the no-arbitrage paradigm, we define a forward swap probability measure such that the forward swap rates $S_{n,N}(t)$ are martingale. For the forward swap rate, the most convenient choice of numeraire is the PVBP, $P_{n+1,N}(t)$. The PVBP is a portfolio of traded assets that has a strictly positive value, therefore we can use it as a numeraire[6]. Let P^{T_{n+1},T_N} denote the forward swap measure associated with the numeraire $P_{n+1,N}(t)$. Then the denominated price processes, $\{S_{n,N}(t)\} = \left\{\frac{v_n(t)-v_N(t)}{P_{n+1,N}(t)}\right\}$ are martingales under P^{T_{n+1},T_N}. This is summarised in the lemma below.

Lemma 4.1- *Let $S_{n,N}(t)$ denote the time t value of the forward swap rate that starts at time $T_n,\ n = 1, 2, ..., N-1$, and ends at time T_N, where $T_N - T_n$ is the swap tenor. Then the forward swap rate is a martingale under the forward swap measure P^{T_{n+1},T_N} .*

Proof

From the lemma above we need to show that $E_t^{n+1,N}[S_{n,N}(s)] = S_{n,N}(t) \quad t < s \leq T_N$, where the choice of numeraire is the PVBP, $P_{n+1,N}(t)$. Now

$$E_t^{n+1,N}[S_{n,N}(s)] \quad = \quad E_t^{n+1,N}\left[\frac{v_n(s) - v_N(s)}{P_{n+1,N}(s)}\right].$$

[5]Note this is the same SDE for bonds discussed in Chapter 3 given by equation (3.4).
[6]Refer to Chapter 2, subsection 2.6.1 where we define a numeraire and its properties.

But the process $\left\{\frac{v_n(t)-v_N(t)}{P_{n+1,N}(t)}\right\}$ is martingale under P^{T_{n+1},T_N}, therefore

$$E_t^{n+1,N}[S_{n,N}(s)] = \frac{v_n(t)-v_N(t)}{P_{n+1,N}(t)} = S_{n,N}(t),$$

which implies that the forward swap rate $S_{n,N}(t)$ is a martingale under the forward swap measure, P^{T_{n+1},T_N}.

Thus it may be plausible to assume that the process $\{S_{n,N}(t)\}$ follows the SDE[7]

$$\frac{dS_{n,N}(t)}{S_{n,N}(t)} = \gamma_{n,N}(t)dz^{n+1,N}, \quad n = 1, 2, ...N-1, \tag{4.12}$$

where $z^{n+1,N} \equiv \{z^{n+1,N}(t)\}$ is a standard Brownian motion under P^{T_{n+1},T_N} for some volatility $\gamma_{n,N}(t)$.

In order to complete the construction of the log-normal SMM, we have to ensure that the forward swap rate $S_{n,N}(t)$ follow a log-normal process. This is the assumption made for the Black-Scholes formula for swaptions. This transformation can be achieved by using Ito's formula (Theorem 2.1) by determining the log-price of the forward swap rate. This is shown below.

Using Ito's formula we want to determine the SDE of the process $Y(t) = log\big(S_{n,N}(t)\big)$, where the process $\{S_{n,N}(t)\}$ is a solution to equation (4.12). From the Ito's formula we note that $f(x,t) = log(x)$, if and only if , $x = S_{n,N}(t)$, thus

$f_t(x,t) = 0$, $\quad f_x(x,t) = \frac{1}{x} = \frac{1}{S_{n,N}(t)}$, $\quad f_{xx}(x,t) = -\frac{1}{x^2} = -\frac{1}{S^2_{n,N}(t)}$, $\quad \mu(x,t) = 0$ and $\sigma(x,t) = S_{n,N}(t)\gamma_{n,N}(t)$.

Thus

$$dlog\Big(S_{n,N}(t)\Big) = [f_t(x,t) + f_x(x,t)\mu(x,t) + \frac{1}{2}f_{xx}(x,t)\sigma^2(x,t)]dt + [f_x(x,t)\sigma(x,t)]dz^{n+1,N}.$$

It then follows from substitution into the above equation that the log-price of the forward swap rate, $S_{n,N}(t)$ follows the SDE

$$dlog\Big(S_{n,N}(t)\Big) \quad = \quad -\frac{1}{2}\gamma^2_{n,N}(t)dt + \gamma_{n,N}(t)dz^{n+1,N}, \tag{4.13}$$

which can be written in its integral form as

$$log(S_{n,N}(T_n)) - log(S_{n,N}(t)) \quad = \quad -\int_t^{T_n} \frac{1}{2}\gamma^2_{n,N}(s)ds + \int_t^{T_n} \gamma_{n,N}(s)dz^{n+1,N}.$$

[7]Refer to Proposition 2.3 and Proposition 2.4 in subsection 2.5.3.

Therefore

$$log\left[\frac{S_{n,N}(T_n)}{S_{n,N}(t)}\right] = -\int_t^{T_n}\frac{1}{2}\gamma_{n,N}^2(s)ds + \int_t^{T_n}\gamma_{n,N}(s)dz^{n+1,N}, \qquad (4.14)$$

which can be expressed as an explicit solution of $S_{n,N}(T_n)$ given by

$$S_{n,N}(T_n) = S_{n,N}(t)exp\left[-\int_t^{T_n}\frac{1}{2}\gamma_{n,N}^2(s)ds + \int_t^{T_n}\gamma_{n,N}(s)dz^{n+1,N}\right]. \qquad (4.15)$$

Thus from the above expression we can conclude that the process $\{S_{n,N}(T_n)\}$ follows a log-normal distribution with mean $\lambda_{n,N}(t) = -\int_t^{T_n}\frac{1}{2}\gamma_{n,N}^2(s)ds$ and variance $\xi_{n,N}^2(t) = \int_t^{T_n}\gamma_{n,N}^2(s)ds$, if the volatility function of the forward swap rate given by $\gamma_{n,N}(t)$ is a deterministic function of time. ■

The mean and variance is derived in a similar way as we did for the SDE of the forward LIBOR rate[8]. Now that we have explored the dynamics of the forward swap rate we are now in a position to state the existence of the SMM. This is stated in the proposition that follows.

Proposition 4.1 *Suppose that the process $\{S_{n,N}(t)\}$ is a solution to the SDE for forward swap rates $S_{n,N}(t),\ n = 1, ..., N-1$, defined by*

$$dS_{n,N}(t) = \mu_{n,N}(t)dt + \gamma_{n,N}(t)dz, \quad n = 1, 2, ...N-1,$$

where $\mu_{n,N}(t)$ is the drift function , $\gamma_{n,N}(t)$ the volatility function and $z = \{z(t)\}$ is a standard Brownian motion under empirical probability P. Then the log-normal Swap Market Model (SMM) exists if and only if,

1. *The dynamics of the forward swap rate $S_{n,N}(t)$ is given by equation* (4.7), *and the initial term structure $S_{n,N}(0)$ is known.*

2. *$S_{n,N}(t)$ is a martingale under forward swap probability measure P^{T_{n+1},T_N}, and the SDE of the log-price of the forward swap rate $S_{n,N}(t)$ is given by equation* (4.13) *, for some volatility $\gamma_{n,N}(t)$ and $z^{n+1,N} = \{z^{n+1,N}(t)\}$ is standard Brownian motion under P^{T_{n+1},T_N}.*

3. *The volatility function $\gamma_{n,N}(t),\ t \leq T_n$, is a deterministic function of time t, which implies that $S_{n,N}(t)$ is log-normally distributed with mean $\lambda_{n,N}(t) = -\int_t^{T_n}\frac{1}{2}\gamma_{n,N}^2(s)ds$ and variance $\xi_{n,N}^2(t) = \int_t^{T_n}\gamma_{n,N}^2(s)ds$.*

Remark: The implication of the above proposition is that each of the SDE's given by equation (4.12) are known as the Swap Market Model for the forward swap rate $S_{n,N}(t)$ under the forward swap measure P^{T_{n+1},T_N}. Brigo, Mercurio and Morini ([5]) cite that the SMM is not compatible

[8]Refer to Section 3.1 in Chapter 3 for this derivation.

with the LMM, since the evolution of the forward swap rates, $S_{n,N}(t)$ under a given numeraire are different within the two models.

4.2 Co-terminal Measure Dynamics of the forward swap rate.

In this section we outline the co-terminal measure dynamics of the forward swap rate, based on the formulation done by Jamshidan ([12]). We assume that the tenor structure of forward swap rates is given by $T_0 < T_1 < ... < T_i < T_{i+1} < ... < T_N$, where the tenor is defined by $\delta_i = T_{i+1} - T_i,\ i = 1, 2, ..., N-1$. Under the co-terminal measure denoted by P^{T_N,T_N}, we consider a forward (payer) swap which starts at time T_n and has $N-n$ accrual periods whose consecutive lengths are given by $\delta_n,\ n = 1, 2, ..., N-1$. Then last settlement date is thus T_N for every swap considered here. It is important to note that the underlying swap agreements differ in length, but they all have a common expiration date T_N. This specific feature justifies the name of co-terminal forward swap rates (Rutkowski, [27]). The figure below is a graphical representation of the tenor structure for co-terminal forward swap rates.

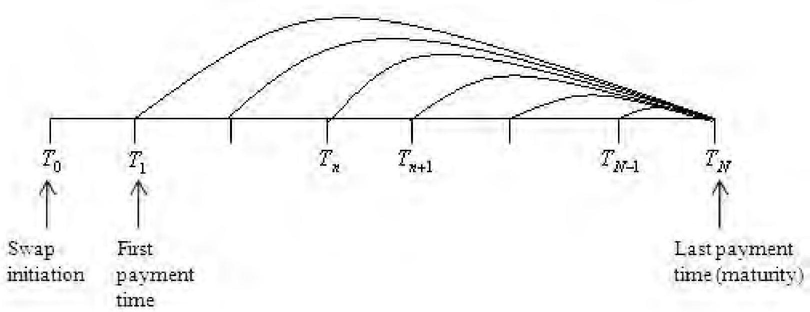

Figure 4.1: Co-Terminal forward swap rates for the co-terminal SMM.

Now consider the forward swap rate $S_{n,N}(t)$ given by equation (4.7). To derive the co-terminal dynamics of the forward swap we shall set for every $1 \le n \le i \le N-1$, the variable

$$g_{ni} \equiv g_{ni,N} = \sum_{k=i}^{N-1} \delta_k \prod_{l=n+1}^{k} (1 + \delta_{l-1} S_{l,N}(t)), \quad g_n \equiv g_{nn}, \quad 1 \le n \le i \le N-1. \tag{4.16}$$

Under this technical condition it follows that the forward swap rates $S_{n,N}(t)$ are martingales under the co-terminal measure P^{T_N,T_N}. Now using the above expression, Jamshidan ([12]) suggests that one can use backward induction to deduce the dynamics of the drift term of the forward swap rates under the co-terminal measure. Its follows after rigorous computations- which are beyond the scope of this thesis- that the drift term takes the form

$$\mu_{n,N}^{co-term}(t) = -S_{n,N}(t)\gamma_{n,N}(t)\sum_{j=n+1}^{N-1}\frac{\delta_{j-1}S_{j,N}(t)\gamma_{j,N}^{'}(t)}{(1+\delta_{j-1}S_{j,N}(t))}\frac{g_{ni}}{g_n}.$$

The next proposition outlines the dynamics of the forward swap rates under the co-terminal measure (see [12]) .

Proposition 4.2 *Let $S_{n,N}(t)$ denote the forward swap rate contracted for period $T_N - T_n$, $n = 1, 2, ..., N-1$, with volatility function $\gamma_{n,N}(t)$. Then under the co-terminal measure P^{T_N,T_N} the forward swap rates are martingales following the SDE*

$$dS_{n,N}(t) = \mu_{n,N}^{co-term}(t)dt + \gamma_{n,N}(t)S_{n,N}(N)dz^{N,N}, \quad n = 1, 2, ...N-1,$$

where

$$\mu_{n,N}^{co-term}(t) = -S_{n,N}(t)\gamma_{n,N}(t)\sum_{j=n+1}^{N-1}\frac{\delta_{j-1}S_{j,N}(t)\gamma_{j,N}^{'}(t)}{(1+\delta_{j-1}S_{j,N}(t))}\frac{g_{ni}}{g_n},$$

and the g_{ni}'s are determined using equation (4.16) *and $z^{N,N} \equiv \{z^{N,N}(t)\}$ is a standard Brownian motion under P^{T_N,T_N}*

Remarks:(see [12] and [27])

1. The forward LIBOR rates and forward swap rates satisfy the following relationship

$$S_{n,N}(t) = \frac{\prod_{j=i}^{N-1}(1+\delta_j L_j(t)) - 1}{\sum_{j=1}^{N-1}\delta_j \prod_{k=j+1}^{N-1}(1+\delta_k L_k(t))}. \tag{4.17}$$

2. The LIBOR and swap market models are inconsistent with each other, that is the forward LIBOR and swap rates cannot simultaneously have deterministic volatilities.

4.3 Pricing of Swaptions.

In this section we use the SMM constructed in section 4.1 to determine the time t arbitrage-free price of a swaption. We show that swaption price constructed within the SMM has the same structure as that of the Black-Scholes price for swaptions. Firstly we define a swaption, and then we develop the Black-Scholes formula for a swaption. Lastly we construct the arbitrage-free price of swaptions using the SMM.

Definition 4.4: Swaption

Let $S_{n,N}(t) \equiv S(t, T_n, T_N)$ denote the time t forward swap rate that starts at time T_n, $n = 1, 2, ..., N-1$, and ends at time T_N . Then an interest rate derivative whose payoff at time T_n is given by

$$\{V_{n,N}^{PS}(T_n)\}^+ \equiv \sum_{i=n+1}^{N} \delta_{i-1} v_i(T_n) \Big\{ S_{n,N}(T_n) - K \Big\}_+, \tag{4.18}$$

where

$$\{x\}_+ = max\{x, 0\} = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

and $\sum_{i=n+1}^{N} \delta_{i-1} v_i(T_n) = P_{n+1,N}(T_n)$, is known as a *payer swaption.*

From the above definition we note that a swaption is simply a call option written on the swap rate $S_{n,N}(t)$. To be more precise it is a financial contract that gives the holder the right but not the obligation to enter into an interest rate swap that starts at time T_n and ends at time T_N, where the holder pays a fixed rate and receives a floating rate. We denote the time t price of the payer swaption as $PSwp(t, T_n, T_N) \equiv PSwp_{n,N}(t)$. Now in order to determine the arbitrage-free price of the payer swaption at time t, we construct its price under the forward swap measure P^{T_{n+1}, T_N}, with associated numeraire, the present value of a basis point (PVBP) denoted $P_{n+1,N}(t)$.

Now consider that the payer swaption is replicated through some self-financing portfolio, then the discounted price process $\{PSwp_{n,N}^{n+1,N}(t)\} = \left\{ \frac{PSwp_{n,N}(t)}{P_{n+1,N}(t)} \right\}$, is a martingale under the swap measure P^{T_{n+1}, T_N}. Hence the time t price of the payer swaption is given by

$$PSwp_{n,N}^{n+1,N}(t) = E_t^{n+1,N} \left[\frac{P_{n+1,N}(T_n) \Big\{ S_{n,N}(T_n) - K \Big\}_+}{P_{n+1,N}(T_n)} \right].$$

Therefore

$$PSwp_{n,N}(t) = P_{n+1,N}(t) E_t^{n+1,N} \Big[\Big\{ S_{n,N}(T_n) - K \Big\}_+ \Big],$$

which can be expressed as

$$PSwp_{n,N}(t) = \sum_{i=n+1}^{N} \delta_{i-1} v_i(t)\, E_t^{n+1,N} \Big[\Big\{ S_{n,N}(T_n) - K \Big\}_+ \Big], \tag{4.19}$$

since $P_{n+1,N}(t) = \sum_{i=n}^{N-1} \delta_i v_{i+1}(t) \equiv \sum_{i=n+1}^{N} \delta_{i-1} v_i(t)$.

Remark: From the above expression we can subsequently derive the Black-Scholes formula for a payer swaption. In the next subsection we state the Black-Scholes formula for a payers swaption, with reference to the Black-Scholes formula for a European call option discussed in chapter 2, section 2.8.

4.3.1 Pricing swaptions using the Black-Scholes Formula.

Proposition 4.3 Black Scholes formula for a payer swaption.

Let $PSwp_{n,N}(t)$ denote the time t price of a payer swaption. Then the time t Black-Scholes price of a payer swaption, denoted by $PSwp_{n,N}^{BLACK}(t)$, written on the forward swap rate $S_{n,N}(t)$, starting at time T_n, $n = 1, ..., N-1$ and maturing at time T_N with strike swap rate K is given by

$$PSwp_{n,N}^{BLACK}(t) = \sum_{i=n+1}^{N} \delta_{i-1} v_i(t) \Big[S_{n,N}(t)\Phi(d) - K\Phi\big(d - \sigma_{n,N}\sqrt{T_n - t}\big) \Big], \tag{4.20}$$

where

$$d = \frac{log\left(\frac{S_{n,N}(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2(T_n - t)}{\sigma_{n,N}\sqrt{T_n - t}} \quad and \quad \sum_{i=n+1}^{N} \delta_{i-1} v_i(t) = P_{n+1,N}(t) \ (PVBP).$$

The function $\sigma_{n,N}$ denotes the volatility for the forward swap rate $S_{n,N}(t)$ and $\Phi(d)$ is the distribution function of the standard normal distribution.

Remark: The above stated formula can be easily verified using the Black-Scholes formula for European call options given in Theorem 2.7 in section 2.8. This is achieved by setting $c(S,t) = PSwp_{n,N}^{BLACK}(t)$, $S(t) = S_{n,N}(t)$, $\sigma\sqrt{T-t} = \sigma_{n,N}\sqrt{T_n - t}$ and $B(t) = P_{n+1,N}(t)$.

4.3.2 Pricing swaptions using the Swap Market Model (SMM).

In this section we show how to price payer swaptions under the SMM. We show that the price of a payer swaption coincides with that of the Black-Scholes formula for swaption equation (4.20), in structure. Now under the no-arbitrage paradigm the time t price of the payer swaption is given by equation (4.19). From this payoff function we are able to construct the SMM price of a payer swaption, using the dynamics of the SMM developed in section 4.1. The time t price of a payer swaption calculated using the SMM is given in the proposition below.

Proposition 4.4 SMM price of a payers swaption.

Consider the payoff function of the payer swaption at time T_n accrued for the time interval $[T_n, T_N]$, $n = 1, ..., N-1$, given by

$$\{V_{n,N}^{PS}(T_n)\}^+ \equiv \sum_{i=n+1}^{N} \delta_{i-1} v_i(T_n) \Big\{ S_{n,N}(T_n) - K \Big\}_+ .$$

Then under the Swap Market Model (SMM), the time t price of the payer swaption, denoted by $PSwp_{n,N}^{SMM}(t)$, written on the forward swap rate $S_{n,N}(t)$ with maturity time T_N and strike swap rate K is given by

$$PSwp_{n,N}^{SMM}(t) = \sum_{i=n+1}^{N} \delta_{i-1} v_i(t) \Big[S_{n,N}(t)\Phi(d) - K\Phi(d - \xi_{n,N}(t)) \Big], \tag{4.21}$$

where

$$d = \frac{log\Big(\frac{S_{n,N}(t)}{K}\Big) + \frac{1}{2}\xi_{n,N}^2(t)}{\xi_{n,N}(t)},$$

and $\xi_{n,N}(t) = \sqrt{\int_t^{T_n} \gamma_{n,N}^2(s)ds}$ denotes the volatility function of the forward swap rate $S_{n,N}(t)$.

Proof

Consider the time t price of an arbitrage-free payer swaption evaluated within the SMM, similar to equation (4.19), such that

$$PSwp_{n,N}^{SMM}(t) = P_{n+1,N}(t) E_t^{n+1,N} \Big[\Big\{ S_{n,N}(T_n) - K \Big\}_+ \Big],$$

where $P_{n+1,N}(t) = \sum_{i=n}^{N-1} \delta_i v_{i+1}(t) \equiv \sum_{i=n+1}^{N} \delta_{i-1} v_i(t)$, and $E_t^{n+1,N}$ denotes that conditional expectation under the forward swap measure P^{T_{n+1},T_N}.

From equation (4.15), $S_{n,N}(T_n)$ can be explicitly expressed as

$$S_{n,N}(T_n) = S_{n,N}(t) exp\Bigg[-\int_t^{T_n} \frac{1}{2}\gamma_{n,N}^2(s)ds + \int_t^{T_n} \gamma_{n,N}(s) dz^{n+1,N} \Bigg],$$

where the forward swap rate $S_{n,N}(T_N)$ is log-normally distributed with mean $\lambda_{n,N}(t) = -\int_t^{T_n} \frac{1}{2}\gamma_{n,N}^2(s)ds$ and variance $\xi_{n,N}^2(t) = \int_t^{T_n} \gamma_{n,N}^2(s)ds$.

Therefore the time t price of the payer swaption is given by

$$PSwp_{n,N}(t) = P_{n+1,N}(t)E_t^{n+1,N}\Big[\Big\{S_{n,N}(t)e^{-\int_t^{T_n}\frac{1}{2}\gamma_{n,N}^2(s)ds+\int_t^{T_n}\gamma_{n,N}(s)dz^{n+1,N}} - K\Big\}_+\Big].$$

Now under the SMM, the volatility function $\gamma_{n,N}(t)$ is assumed to be a deterministic function of time, thus we have that

$$\int_t^{T_n} \gamma_{n,N}(s)dz^{n+1,N} \sim N\Big(0, \int_t^{T_n} \gamma_{n,N}^2(s)ds\Big),$$

therefore

$$\xi_{n,N}(t)N(0,1) \sim N\Big(0, \int_t^{T_n} \gamma_{n,N}^2(s)ds\Big),$$

where $\xi_{n,N}(t) = \sqrt{\int_t^{T_n} \gamma_{n,N}^2(s)ds}$.

Simplifying notation, let $\xi_{n,N} = \xi_{n,N}(t)$ and let the random variable $Y \sim N(0,1)$. Therefore the present value of the payer swaption is given by

$$\begin{aligned} PSwp_{n,N}(t) &= P_{n+1,N}(t)E_t^{n+1,N}\Big[\Big\{S_{n,N}(t)e^{-\frac{1}{2}\xi_{n,N}^2+\xi_{n,N}Y} - K\Big\}_+\Big] \\ &= P_{n+1,N}(t)\Big[\int_{-\infty}^{\infty}\Big\{S_{n,N}(t)e^{-\frac{1}{2}\xi_{n,N}^2+\xi_{n,N}\,y} - K\Big\}_+ \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\Big]. \end{aligned}$$

Now at each to maturity time, $T_n,\ n = 1, ..., N-1$, the holder of the payer swaption will exercise the option if and only if $S_{n,N}(T_n) \geq K$, which implies that

$$\begin{aligned} S_{n,N}(T_n) &\geq K \\ S_{n,N}(t)e^{-\frac{1}{2}\xi_{n,N}^2+\xi_{n,N}\,y} &\geq K. \end{aligned}$$

Therefore

$$y = \frac{log\Big(\frac{K}{S_{n,N}(t)}\Big) + \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}}.$$

Now if we let $\omega_1 = \frac{log\left(\frac{K}{S_{n,N}(t)}\right)+\frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}}$, then the present value of the payer swaption is given by

$$\begin{aligned}
PSwp_{n,N}(t) &= P_{n+1,N}(t)\Big[\int_{\omega_1}^{\infty}\Big\{S_{n,N}(t)e^{-\frac{1}{2}\xi_{n,N}+\xi_{n,N}\,y} - K\Big\}_+ \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy\Big] \\
&= P_{n+1,N}(t)\left[S_{n,N}(t)\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}+\xi_{n,N}\,y-\frac{1}{2}\xi_{n,N}}\,dy - K\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy\right] \\
&= P_{n+1,N}(t)\left[S_{n,N}(t)\int_{\omega_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\xi_{n,N})^2}\,dy - K(1-\Phi(\omega_1))\xi_{n,N}\right].
\end{aligned}$$

Also if we let $x = y - \xi_{n,N}$ and $\omega_2 = \omega_1 - \xi_{n,N}$, then

$$\begin{aligned}
PSwp_{n,N}(t) &= P_{n+1,N}(t)\left[S_{n,N}(t)\int_{\omega_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx - K(1-\Phi(\omega_1))\right] \\
&= P_{n+1,N}(t)\Big[S_{n,N}(t)(1-\Phi(\omega_2)) - K(1-\Phi(\omega_1))\Big] \\
&= P_{n+1,N}(t)\Big[S_{n,N}(t)\Phi(-\omega_2) - K\Phi(-\omega_1)\Big],
\end{aligned}$$

where

$$\begin{aligned}
\Phi(-\omega_2) &= \Phi(-\omega_1 + \xi_{n,N}) \\
&= \Phi\left(\frac{-log\left(\frac{K}{S_{n,N}(t)}\right) - \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}} + \xi_{n,N}\right) \\
\Phi(-\omega_2) &= \Phi\left(\frac{log\left(\frac{S_{n,N}(t)}{K}\right) + \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\Phi(-\omega_1) &= \Phi\left(\frac{-log\left(\frac{K}{S_{n,N}(t)}\right) - \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}}\right) \\
\Phi(-\omega_1) &= \Phi\left(\frac{log\left(\frac{S_{n,N}(t)}{K}\right) + \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}} - \xi_{n,N}\right).
\end{aligned}$$

Substituting the above leads to the time t value of a payer swaption priced using the SMM, and is given by

$$PSwp_{n,N}^{SMM}(t) = \sum_{i=n+1}^{N} \delta_{i-1} v_i(t) \Big[S_{n,N}(t)\Phi(d) - K\Phi(d - \xi_{n,N}) \Big],$$

where

$$d = \frac{log\Big(\frac{S_{n,N}(t)}{K}\Big) + \frac{1}{2}\xi_{n,N}^2}{\xi_{n,N}},$$

$\xi_{n,N} = \xi_{n,N}(t) = \sqrt{\int_t^{T_n} \gamma_{n,N}^2(s)ds}$ denotes the volatility function of the forward swap rate $S_{n,N}(t)$ and $\Phi(d)$ is the distribution function of the standard normal distribution. ■

Remark: From the above we can observe that the SMM price of a payer swaption has a similar structure to that of the Black-Scholes formula if and only if $\sigma_{n,N}\sqrt{T_n - t} = \xi_{n,N}(t)$, that is $\sigma_{n,N} = \frac{1}{\sqrt{T_n - t}}\sqrt{\int_t^{T_n} \gamma_{n,N}^2(s)ds}$. The consequence of this assumption is that the SMM can be easily calibrated to calculate swaption prices by substituting the SMM volatility function with Black-Scholes implied volatilities of the swaptions. This leads to a pricing formula similar to the Black-Scholes pricing formula for swaptions.

In conclusion, in chapter 4 we constructed the arbitrage-free log-normal SMM for pricing swaptions, whose underlying asset is the forward swap rate $S_{n,N}(t)$. We have shown that the SMM is defined by the set of SDE's given by equation (4.13), where $z^{n+1,N} = \{z^{n+1,N}(t)\}$ is a standard Brownian motion under forward swap measure P^{T_{n+1},T_N}. We have also discussed the co-terminal SMM, and we observe that forward swap rates will not be martingale under the co-terminal probability measure P^{T_N,T_N}. This shows that the LMM and SMM are incompatible models. In the next chapter we perform comparative numerical analysis of the "market models" with the Black-Scholes model for pricing interest-rate derivatives. This will be done using Monte Carlo methods.

Chapter 5

Methodology and Analysis

In Chapters 3 and 4 we constructed the arbitrage-free, log-normal models i.e. the LMM and SMM for pricing caplets and swaptions, respectively using SDE's. In this chapter we implement the LMM and SMM for pricing the respective interest rate derivatives using Monte Carlo methods. Now given the complexity of the stochastic processes of the LMM and SMM under their respective martingale measures, it is not easy to solve them explicitly. Hence, if we want to price caplets and swaptions within these market models , we need use numerical methods. One method which is widely used for market models is Monte Carlo simulation. The aim in this section is to thus illustrate the use such numerical methods. It is important to note that the dynamics of the LMM and SMM were constructed in the continuous-time framework, but in order to perform Monte Carlo simulation within these models we need to transform them into their discrete-time counterparts. Such a transformation reduces the computational burden as we only need to simulate the SDE's describing the forward rate (LIBOR or swap) dynamics for a finite number of maturities. The simplest and effective method for discretization of SDE's is the Euler-Maruyama method. This will be outlined in the section that follows. First we describe the idea behind the Monte Carlo simulation, and then we give the theoretical implication to the LMM and SMM.

5.1 Monte Carlo Simulation

Monte Carlo simulation is a method for iteratively evaluating a deterministic model using sets of random numbers as inputs. This method is often used when the model is complex, nonlinear, or involves more than just a couple of uncertain parameters. At the core of Monte Carlo simulation is the generation of random numbers. Several techniques can be used to generate random numbers (see [26]), but for this thesis in particular, random number generation will be achieved using built-in functions of the statistical software MATLAB. Once we have generated random numbers, Monte Carlo techniques can be used to estimate the expected value of some random variables. Before we proceed to implementation of Monte Carlo simulation for the LMM and SMM , we need to state some crucial limit theorems on which this method is built on and they are given below[1].

Theorem 5.1 Strong law of large numbers.

For a family of independent and identically distributed (i.i.d) random variables $X_1, X_2, ..., X_n$ *suppose that the mean* $\mu = E[X_i]$ *exists. Then,*

$$\lim_{n \to \infty} \frac{X_1 + X_2 + ... + X_n}{n} = \mu,$$

with probability one.

Remark: The strong law of large numbers ensures that the sample mean $\bar{X_n} = E[X] = \frac{1}{n}(X_1 + X_2 + ... + X_n)$ converges to the unknown population mean μ almost surely as $n \to \infty$. In our case, the average $E[X]$ is often called the Monte Carlo expectation of the random variable X.

Theorem 5.2 Central Limit Theorem

Let $X_1, X_2, ..., X_n$ *be i.i.d random variables with* $E[X_i] = \mu$ *and* $V[X_i] = \sigma^2 < \infty$. *Define*

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}, \quad \textit{where} \ \ \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then the distribution function of Z_n *converges to the standard normal distribution function as* $n \to \infty$*, that is,*

$$\lim_{n \to \infty} P(Z_n \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

Remark: The central limit theorem implies that probability statements about Z_n can be approximated by corresponding probabilities for the standard normal random variable if n is large, usually for values $n > 30$.

Now in the context of financial engineering, Monte-Carlo simulation is a technique used for the numerical realization of a stochastic process by means of normally distributed random variables. It can then be used for the computation of the expected future value of option prices in situations where explicit solutions are not accessible. In particular we can determine the prices of caplets within the LMM by expressing them in the form of their future expected discounted payoff function under the martingale measure (terminal measure) P^{T_N} as given in equation (3.20). Similarly we can determine swaption prices within the SMM by expressing them in terms of their future expected discounted payoff under the forward swap martingale measure P^{T_{n+1},T_N},

[1]Refer to any text in mathematical statistics for proof and justification of these theorems.

as given in equation (4.19). Once we have expressed our interest-rate derivatives in terms of expected future payoffs what is left is to calculate a quantities of the form $E[X]$, where X is a random variable representing the payoff functions for caplets and swaptions. This is when we implement Monte-Carlo methods to evaluate the expected payoffs by generating random variables under the distribution of X, and if the sample size n is large enough , it follows from the Strong law of large numbers that we approximate $E[X]$ with

$$E[X] \approx \frac{1}{n}\sum_{i=1}^{n} X_i.$$

As mentioned earlier, in order to determine the numerical solutions of the continuous-time market models we need to first transform them into their discrete-time counterparts. This can be achieved using the Euler-Maruyama method. We define the Euler-Maruyama method (see [17]) for a general SDE as the one developed in Chapter 2[2].

5.1.1 Euler-Maruyama (Euler) Method.

Let $\{X(t)\ t \geq 0\}$ be a diffusion process defined by SDE

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dz, \quad 0 \leq t \leq T, \tag{5.1}$$

where $X(0)$ has a known distribution.

The Euler or Euler-Maruyama method for solving SDE's is a simple generalization of the Euler's method for solving ordinary differential equations. A process $\{X(t),\ t \in [0, T]\}$ may be simulated by partitioning the interval $[0, T]$ into M sub-intervals with step size or length $h = \frac{T}{M}$ and then replacing the SDE with the stochastic difference equation[3]

$$\hat{X}_{k+1} = \hat{X}_k + \mu(\hat{X}_k, kh)h + \sigma(\hat{X}_k, kh)\sqrt{h}Z_k, \tag{5.2}$$

where $Z_1, Z_2, ... \sim_{i.i.d} N(0,1)$. The time series $\{\hat{X}_k,\ k = 0, 1, 2, ..., M\}$ approximates the process $\{X(t),\ t \geq 0\}$, i.e. $\hat{X}_k \approx X(kh),\ k = 0, 1, 2, ..., M$.

Now consider the case when the stochastic process $\{X(t)\ t \geq 0\}$ is a geometric Brownian motion with SDE

$$\frac{dX(t)}{X(t)} = \mu(X(t), t)dt + \sigma(X(t), t)dz, \quad 0 \leq t \leq T. \tag{5.3}$$

[2]Refer to equation (2.7) in subsection 2.5.3 of Chapter 2 for the general definition of a SDE.
[3]Here we use hats to distinguish the discretized variables.

Then the stochastic difference equation describing the path of $X(t)$ is given by

$$\hat{X}_{k+1} = \hat{X}_k exp\Big[\Big(\mu(\hat{X}_k, kh) - \frac{1}{2}\sigma^2(\hat{X}_k, kh)\Big)h + \sigma(\hat{X}_k, kh)\sqrt{h}Z_k\Big], \quad 0 \leq t \leq T, \quad k = 0, 1, 2, ..., M. \tag{5.4}$$

Below is an algorithm for the Euler Method for generating a stochastic process following a geometric Brownian motion.

Algorithm 5.1 Euler Method.

1. *Generate* X_0 *from the distribution of* $X(0)$. *Set* $k = 0$.
2. *Generate* $Z_k \sim N(0, 1)$.
3. *Evaluate* $\hat{X}_{k+1}$ *from equation* (5.4) *as an approximation to* $X(kh)$.
4. *Set* $k = k + 1$ *and go to step 2, and so on.*

Now that we have given a general outline of Monte Carlo method , we are now in a position to implement these techniques to pricing interest derivatives within the LMM and SMM.

5.2 Pricing Interest-rate Derivatives via Monte Carlo simulation.

In this section we outline how to implement Monte Carlo methods to price interest rate derivatives such as caplets and swaptions within the LMM and SMM respectively. A general procedure for pricing European style options using Monte Carlo methods is given below ((see [17])).

1. Simulate n paths of the Brownian motion which is the stochastic driver for the continuous-time securities model.
2. Assuming that we are given the SDE for the underlying asset under martingale measure $\widetilde{P}$, from which the options value is derived, we simulate n sample paths of asset prices over the relevant time interval, say $[0, T]$ by applying a numerical scheme, such as Euler's method.
3. Evaluate the discounted payoff of each asset on each sample path, as determined by the specifics of the asset.
4. Compute a Monte Carlo estimate of the theoretical option value using the n discounted cash flows over the sample paths.

Having listed the general procedure to pricing interest-rate derivatives using Monte-Carlo methods, we can apply these four steps in order to price caplets and swaptions and this will be discussed in the section that follows.

5.2.1 Simulation of caplet prices within the LMM.

Step 1- Simulating the path of the standard Brownian motion $z^{i+1} = \{z^{i+1}(t)\}$.

In section 3.2 of Chapter 3 we established that under the terminal measure P^{T_N}, $z^{i+1} = \{z^{i+1}(t)\}$ is a standard Brownian motion (s.b.m.). Now from the definition of standard Brownian motion[4], it follows that the increments $z^{i+1}(t + \Delta t) - z^{i+1}(t)$ are independent random variables which are normally distributed with mean 0 and variance Δt. The path for the standard Brownian motion z^{i+1} can be constructed as follows. Let $0 = T_0 < T_1 < ... < T_k < ... < T_N$, be a set discretized time intervals for which the path of the s.b.m. follows. Let $\Delta T = T_{k+1} - T_k = \frac{T_N - T_0}{N}$ be the time step size for the s.b.m. for $k = 0, 1, ..., N$. Now assuming that the initial value is given by $z^{i+1}(T_0) = 0$, we can then simulate the path of the s.b.m. using the following algorithm.

Algorithm 5.2 Simulating standard Brownian motion path.

1. *Start with initial value* $z^{i+1}(0) = 0$, *and step size* $\Delta T = \frac{T_N - T_0}{N}$.
2. *Generate random variables* $\varepsilon_k \sim N(0, 1)$ *for* $k = 1, ..., N$, *using the* **randn** *function in MATLAB.*
3. *Generate the path of the s.b.m. using the expression*

$$z^{i+1}(T_{k+1}) = z^{i+1}(T_k) + \sqrt{\Delta T}\varepsilon_k, \quad k = 1, ..., N. \tag{5.5}$$

Remark: This algorithm returns the discrete path of a s.b.m. which is exact in the sense that the $z^{i+1}(T_k)$ are drawn from their respective distribution, in this case, the standard normal distribution.

Step 2- Simulation of the LIBOR Market Model (LMM).

In step 1 we have the simulated path of the s.b.m. Next we generate the path of the forward LIBOR rates within the LMM. For this we shall use algorithms similar those suggested by Glasserman et.al ([9]). Consider the tenor structure $T_1 < ... < T_i < T_{i+1} < ... < T_N$, tenor $\delta_i = T_{i+1} - T_i, \quad i = 1, 2, ..., N - 1$, and the time t forward LIBOR rate $L(t, T_i, T_{i+1}) \equiv L_i(t)$ contracted for the period $[T_i, T_{i+1}]$ as given in equation (3.3). Now under the terminal measure P^{T_N}, we established that the forward LIBOR rate is martingale, and that the dynamics of the LMM are described by the SDE given in equation (3.17), i.e

$$dL_i(t) = L_i(t)\mu_i^{term}(t)dt + L_i(t)\gamma_i(t)dz^N, \quad i = 1, .., N - 1,$$

[4] See definition 2.3 in Chapter 2.

where

$$\mu_i^{term}(t) = \begin{cases} -\sum_{j=i+1}^{N-1} \frac{\delta_j L_j(t)\gamma_j^{'}(t)\gamma_i(t)}{1+\delta_j L_j(t)} & \text{for } i < N-1 \\ 0 & \text{for } i = N-1 \end{cases}.$$

Now using Euler's discretization scheme it follows that the forward LIBOR rates $L_i(t),\ i = 1, ..., N-1$, are simulated at times $T_k,\ k = 0, 1, ..., i$ with starting point being $L_i(T_0)$ and step size ΔT. Using hats to represent the Euler discretized variables, the LMM can be expressed by the following stochastic difference equation

$$\hat{L}_i(T_{k+1}) = \hat{L}_i(T_k)exp\left[\left(\mu_i^{term}(T_k) - \frac{1}{2}\gamma_i(T_k)\gamma_i(T_k)'\right)\Delta T + \gamma_i(T_k)\sqrt{\Delta T}\varepsilon_k\right], \tag{5.6}$$

where

$$\mu_i^{term}(T_k) = -\sum_{j=i+1}^{N-1} \frac{\delta_j \hat{L}_j(T_k)\gamma_j^{'}(T_k)\gamma_i(T_k)}{1+\delta_j \hat{L}_j(T_k)}. \tag{5.7}$$

Algorithm 5.3 Simulating forward LIBOR rates within the LMM.

1. *Simulate the path of the s.b.m. under the respective terminal measure as in Step 1.*
2. *Start with initial value* $\hat{L}_i(T_0)$ *of the forward LIBOR rate, and respective volatility function* $\gamma_i(T_0)$*, assumed known.*
3. *For* $k = 0, 1, ..., i$ *and* $i = 1, 2, ..., N-1$*, generate the paths for the discretized forward LIBOR rates,* $\hat{L}_i(T_{k+1})$ *using equation* (5.6).

Remarks: Even though the forward LIBOR rates are martingales with respect to forward measure, after discretization, they lose their martingale property. Therefore LIBOR rates are not martingale under forward measure (Kim and Gaddam, [15]). Glasserman et.al ([9]) discuss the methods of martingale discretization using the discount bonds as the underlying asset, and then recovering the forward LIBOR rates from these bond prices. For simulation purposes, this thesis will not implement martingale discretization as we do not have any market data on bond prices. So once we simulate the forward LIBOR rates $\hat{L}_i(T_k)$ we will be in a position to calculate the caplet payoffs within the LMM.

Step 3-Simulating numeraire prices.

As noted in Chapter 3, in order to evaluate the LMM under the no-arbitrage paradigm we have to find an equivalent probability measure under which the discounted forward LIBOR rates are martingales. Our choice of numeraire under the terminal measure P^{T_N} is the bond with the longest maturity $v(t, T_N) \equiv v_N(t)$. As a result we need to determine these set of bond prices using simulation techniques. Once we determine the default-free discount bond prices we be in a position to evaluated the numeraire based caplet payoffs, given by equation (3.20), i.e.

$$Cpl_i(t) = \delta_i v_{i+1}(t) E_t^{T_{i+1}} \Big[\{L_i(T_i) - K\}_+ \Big], \quad 0 \leq t \leq T_i.$$

Note that from the above equation the caplet prices are dependent on the prices of the discount bond $v_{i+1}(t)$, that have not yet matured. As a result we establish a methodology for simulating these bond prices. This can be achieved using equation (3.3), i.e.

$$L_i(t) = \frac{1}{\delta_i} \left(\frac{v_i(t)}{v_{i+1}(t)} - 1 \right), \quad i = 1, 2, ..., N-1, \quad t \geq 0.$$

Gaddam([15]) suggests that it is sometimes notationally convenient to extend the definition of $L_i(t)$ beyond the i^{th} tenor date. This is done by setting $L_i(t) = L_i(T_i),\ t > T_i$. At a tenor date T_i the price of any bond $v_n(T_i),\ n = i+1, ..., N-1$, that has not yet matured is given by

$$v_n(T_i) = \prod_{j=i}^{n-1} \left(\frac{1}{(1 + \delta_j L_j(T_i))} \right) = \prod_{j=i}^{n-1} (1 + \delta_j L_j(T_i))^{-1}. \tag{5.8}$$

Thus from Euler's scheme it implies that we can find the discretized bond prices $\hat{v_{i+1}}(T_i)$ that have not matured given in equation (3.20) as

$$\hat{v_{i+1}}(T_i) = \prod_{j=i}^{n-1} (1 + \delta_j \hat{L_j}(T_i))^{-1}. \tag{5.9}$$

The following algorithm outlines the procedure for determining these bond prices.

Algorithm 5.4 Simulating bond prices.

1. *Generate the path of the s.b.m. as in Step 1.*
2. *Generate the paths for the discretized forward LIBOR rates, $\hat{L_i}(T_{k+1})$ as in Step 2.*
3. *Generate the discretized bond prices $\hat{v_{i+1}}(T_i)$, using equation* (5.9).

Step 4- Evaluating numeraire based caplet payoffs.

In step 4 , we calculate the numeraire based caplet payoffs. This step implements the Monte Carlo procedure described in Section 5.1. Consider the time t price of the i^{th} caplet given by equation (3.20). Again using hats to denote the discretized variables, equation (3.20) can be written as

$$\hat{Cpl_i}(t) = \delta_i \hat{v_{i+1}}(T_i) E_t^{T_{i+1}} \Big[\{\hat{L}_i(T_i) - K\}_+ \Big]$$

$$= E_t^{T_{i+1}} \Big[\delta_i \hat{v_{i+1}}(T_i) \{\hat{L}_i(T_i) - K\}_+ \Big]. \quad (5.10)$$

Now if the sample size n is large enough, say $n = 10000$, Monte Carlo method suggests that the future expected value of the caplet is estimated as the sample mean of the caplet at time t, as shown in the expression below

$$\hat{Cpl_i}(t) \approx \frac{1}{n} \sum_{i=1}^{n} \hat{Cpl_i}(t). \quad (5.11)$$

Algorithm 5.5 Generating caplet prices within the LMM.

1. *Generate the path of the s.b.m. as in Step 1.*
2. *Generate the paths for the discretized forward LIBOR rates,* $\hat{L}_i(T_{k+1})$ *as in Step 2.*
3. *Generate the discretized bond prices* $\hat{v_{i+1}}(T_i)$, *as in Step 3 .*
4. *Evaluate numeraire based caplet payoff using equation* (5.10).
5. *Repeat (4) , n=10 000 times.*
6. *Evaluate price of the caplet using equation* (5.11).

5.2.2 Simulation of swaption prices within the LMM.

In this section we describe sets of algorithms used to generate swaption prices within the LIBOR Market Model(SMM). Ideally this section was meant to prices swaptions using the SMM under its terminal measure. But due to the complicated dynamics and expressions of the SMM under co-terminal the reader is referred to Jamshidan([12]) and Glasserman et.al ([9]) for the discretization of the SMM and in turn generation of the forward swap rates under the SMM. As a consequence, results in this section are less explicit than in the previous section. We follow the general procedures highlighted in Section 5.2 on pricing interest-rate derivatives using Monte Carlo methods.

Jamshidan ([12]) and Rutkowski ([27]) highlight that the SMM and LMM are inconsistent models in that the forward swap rates $S_{n,N}(t)$, do not follow log-normal processes within the LMM, even though the forward swap rates are a linear combination of several forward LIBOR rates as shown in equation (4.17). Swaptions can not be priced analytically by the LMM. One can use simulation to obtain exact prices of swaptions, by simulating forward swap rates under the LMM. Below are steps and algorithms outlining the procedure for simulating forward swap rates under the LMM which in turn can be used to price swaptions. Here is important to note that we just highlight the procedure theoretically, but we shall not implement them numerically as it is demanding and beyond the scope of this thesis.

Step 1- Simulating the path of the standard Brownian motion within the LMM dynamics.

- Refer to Algorithm 5.2.

Step 2- Simulating forward LIBOR rates within the LIBOR Market Model (LMM).

- Refer to Algorithm 5.3.

Step 3-Simulating the bond prices, that have not yet matured.

- Refer to Algorithm 5.4.

Step 4- Simulating forward swap rates within the LMM.

In this step we generate forward swap using forward LIBOR rates generated under the LMM. This will achieved using expression (4.17), i.e.

$$S_{n,N}(t) = \frac{\prod_{j=i}^{N-1}(1+\delta_j L_j(t)) - 1}{\sum_{j=1}^{N-1}\delta_j \prod_{k=j+1}^{N-1}(1+\delta_k L_k(t))}.$$

Thus the forward swap rates $S_{n,N}(t)$ $n = 1, 2, ..., N-1$ are simulated at times $T_i,\ i = 0, 1, ..., N-1$ with starting point being $T_0 = 0$ and step size $\Delta T = \frac{T_N - T_0}{N}$. Now using hats to describe the discretized forward swap rates, the expression above can be written as

$$\hat{S_{n,N}}(T_i) = \frac{\prod_{j=i}^{N-1}(1+\delta_j \hat{L_j}(T_i)) - 1}{\sum_{j=1}^{N-1}\delta_j \prod_{k=j+1}^{N-1}(1+\delta_k \hat{L_k}(T_i))} \tag{5.12}$$

The algorithm below describes how to simulate forward swap rates.

Algorithm 5.6 Generating forward LIBOR rates within the LMM.

1. *Generate the paths of the standard Brownian motion using algorithm 5.2.*
2. *Generate the paths for the discretized forward LIBOR rates,* $\hat{L_i}(T_{k+1})$ *using algorithm 5.3.*
3. *Generate the paths for the discretized bond prices ,* $\hat{v_{i+1}}(T_i)$ *using algorithm 5.4.*
4. *Generate the forward swap rates* $\hat{S_{n,N}}(t)$ *equation* (5.12).

Remarks: Once we have determined the forward swap rates one will be in a position to price swaptions within the LMM. This will not be considered for this thesis.

5.3 Results

5.3.1 Pricing caplets within the LMM.

Before we proceed to the results of the simulation exercise, we need to setup a base scenario. For numerical comparison of our market models, literature suggests (e.g [9]) that the number of variations of methods, scenarios, and instruments one could investigate numerically is limitless. So our objective in this section is not to be exhaustive of all the scenarios, but instead choosing a simple scenario to illustrate the dynamics of our models. Our choice of a base scenario is as follows.

Scenario

We consider pricing a set of caplets contracted for a period of $T_N = 5$ years. We assume an initial value of the forward LIBOR rate to be $L_i(t) = 5\%$. For the non-exact calibration of the LMM we use a time-homogeneous volatility function , that is constant over time and over individual caplet maturities. Our choice is $\gamma_i(t) = \gamma = 0.2$, taking note this volatility is an intuitive guess. Using a 6-month tenor for the forward LIBOR rate, i.e. $\delta_i = [T_{i+1} - T_i] = 0.5$, the forward LIBOR rates $L_i(t)$, will be reset every 6 months for the 5 year period. As a result the discretized LMM, will have $N = 10$ sub-intervals and a time step of $\Delta T = \frac{T_N - T_0}{N} = \frac{5}{10} = 0.5$. Lastly, we generate the caplet prices assuming a cap rate of 3% and a notional principal of US$100, when the caplet contract is entered.

Step 1-Simulating the path of the standard Brownian motion.

In step 1 we simulate the path of the standard Brownian motion (s.b.m.) over a 5 year period from which the path of forward LIBOR rates will be determined. We assume an initial value for the s.b.m. as $\{z^N(T_0)\} = 0$. To illustrate the general path of the s.b.m., we simulate it for a 5 year period, with very small time steps , i.e. for small time intervals. We then simulate for the base scenario mentioned above for forward LIBOR rates with semi-annual tenor, i.e. we simulate the path of the s.b.m. with a time step of $\Delta T = 0.5$. Below are the plots of the path of the standard Brownian motion for the two cases. Refer to Appendix 1 for the MATLAB code used for this step.

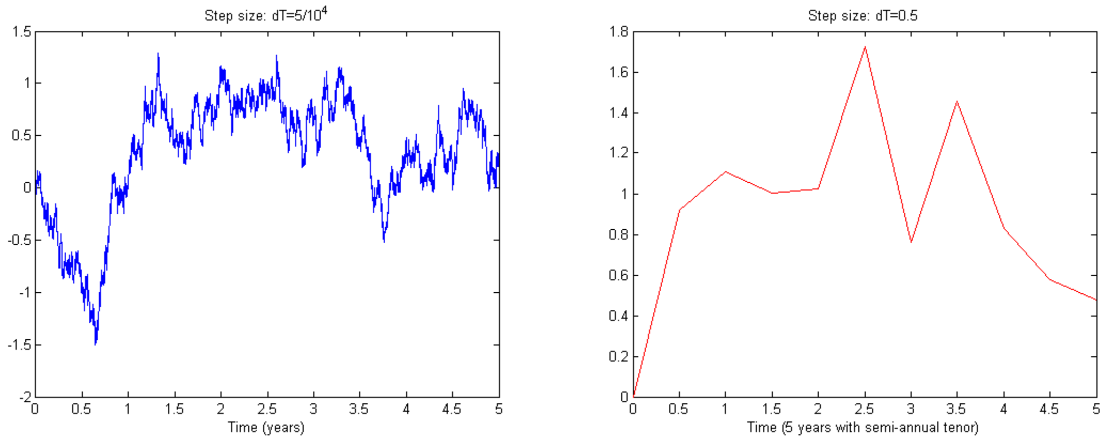

Figure 5.1: Path of a standard Brownian motion.

Step 2: Simulating the forward LIBOR rates under the terminal measure.

In step 2 we simulate the path of the forward LIBOR rates which are the underlying assets for the LMM. This is done within the dynamics of the LMM under the terminal measure P^{T_N} . We simulate the LIBOR rates using the discretized LMM following Euler's scheme as shown in equation (5.6). To achieve this step we assume the initial term structure for the LIBOR rate to be 5% i.e. $L_i(T_0) = 0.05,\ i = 1, 2, ...N-1$, with tenor period $\delta_i = [T_{i+1} - T_i] = 0.5$, which means the LIBOR rates are reset on a 6-month basis (6-month LIBOR rate). The forward LIBOR rates shall be simulated for a 5 year period with a time step of $\Delta T = \frac{T_N - T_0}{N} = \frac{5}{10} = 0.5$. The table below shows the forward LIBOR rates generated for a 6-month LIBOR over a period of 5 years under the terminal measure P^{T_N}. See Appendix 1 for the MATLAB code.

Time (T_k)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$L_0(T_k)$	0.05									
$L_1(T_k)$	0.05	0.0491								
$L_2(T_k)$	0.05	0.0493	0.0487							
$L_3(T_k)$	0.05	0.0489	0.0481	0.0476						
$L_4(T_k)$	0.05	0.0494	0.0489	0.0487	0.0483					
$L_5(T_k)$	0.05	0.0495	0.0492	0.0487	0.0478	0.0474				
$L_6(T_k)$	0.05	0.0498	0.0492	0.0483	0.048	0.0482	0.0477			
$L_7(T_k)$	0.05	0.0487	0.048	0.0473	0.0472	0.0469	0.0466	0.0464		
$L_8(T_k)$	0.05	0.049	0.0483	0.048	0.0467	0.0468	0.046	0.046	0.0453	
$L_9(T_k)$	0.05	0.0495	0.0489	0.0483	0.0476	0.046	0.0464	0.045	0.0461	0.0449

Table 5.1: Forward LIBOR rates “before the last LIBOR” and on the “last LIBOR” settlement dates.

From Table 5.1 we note that the value, say for example, $L_1(T_1) = 0.0491$ represents the forward LIBOR rate contracted for $T = 0.5$ (6 months) to $T = 1$(1 year). The cells with no entries are a result of the fact that the forward LIBOR would have expired for that particular cell. One can also observe that the forward LIBOR are are quite similar to each other. This is a result of the flat-initial term structure of the forward LIBOR rates i.e. $L_i(T_0) = 0.05,\ i = 1, 2, ...N-1$.

The drift matrix was also generated under the terminal measure P^{T_N}, given by equation (5.7) below

$$\mu_i^{term}(T_k) = -\sum_{j=i+1}^{N-1} \frac{\delta_j \hat{L}_j(T_k)\gamma_j'(T_k)\gamma_i(T_k)}{1+\delta_j \hat{L}_j(T_k)}.$$

This drift term corresponds to the forward LIBOR rates "before the last settlement date", i.e. $L_i(T_k)$, $i < N-1$, $k = 0, 1, ..., i$. On the "last settlement date the drift term is equal to zero. The table below shows the drift matrix for the forward LIBOR rates.

Time to maturity	$T_1 = 0.5$	$T_2 = 1$	$T_3 = 1.5$	$T_4 = 2$	$T_5 = 2.5$	$T_6 = 3$	$T_7 = 3.5$	$T_8 = 4$
$T_1 = 0.5$	-0.9756							
$T_2 = 1$	-0.9756	-0.9627						
$T_3 = 1.5$	-0.9756	-0.9552	-0.9385					
$T_4 = 2$	-0.9756	-0.9646	-0.9537	-0.951				
$T_5 = 2.5$	-0.9756	-0.966	-0.9608	-0.9504	-0.9332			
$T_6 = 3$	-0.9756	-0.9711	-0.9601	-0.9439	-0.9379	-0.9415		
$T_7 = 3.5$	-0.9756	-0.9513	-0.938	-0.9251	-0.9217	-0.9167	-0.9117	
$T_8 = 4$	-0.9756	-0.9569	-0.9437	-0.9374	-0.9135	-0.9146	-0.9000	-0.8997

Table 5.2: Drift term for the forward LIBOR rates $(1 * 10^{-3})$ for $i < N-1$.

The plot below shows the forward LIBOR "before" and on the "last" settlement date. For us to see the general path, we plot these forward LIBOR rates for a much smaller time step of say $\Delta T = 0.05$, and then we also construct the same plot for our base scenario with a time step of $\Delta T = 0.5$.

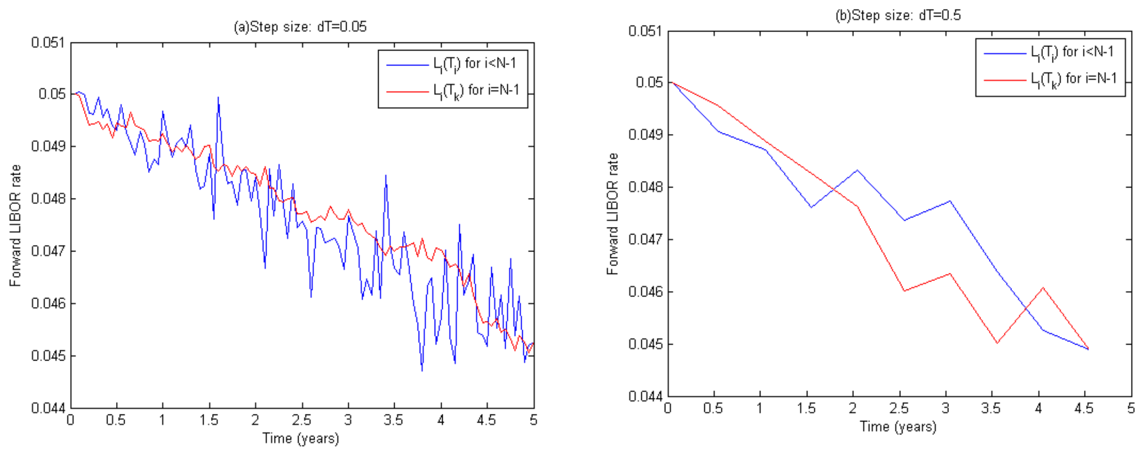

Figure 5.2: Path of the forward LIBOR rate "before" and on "last" settlement date.

From figure 5.2 (a), we can note that the forward LIBOR rates generated "before the last" settlement date, i.e. $i < N-1$, are relatively more volatile as compared those generated on the "last" settlement date for $i = N-1$. This is a result of the drift term of the LMM given by equation (5.7) being a function of the volatility function of the forward LIBOR rates. On the other hand we observe a decreasing trend in the forward LIBOR rates. This is possibly as result of the negative drift term $\mu_i^{term}(T_k)$ and that the next forward LIBOR rate $\hat{L_i}(T_{k+1})$ generated is a dependent on the current value $\hat{L_i}(T_k)$ as shown in equation (5.6).

Step 3-Simulating the numeraire prices.

In step 3 we generated bond prices. This is our choice of numeraire for discounting the forward LIBOR rates under the terminal measure P^{T_N}. The discretized bond prices $\hat{v_{i+1}}(T_i)$ were generated using equation (5.9), and the algorithm 5.4. The results are shown in the table that follows.

Time (T_i)	Bond price ($\hat{v_{i+1}}(T_i)$)
0	1
0.5	0.9756
1	0.9761
1.5	0.9762
2	0.9767
2.5	0.9764
3	0.9769
3.5	0.9767
4	0.9773
4.5	0.9779

Table 5.3: Default-free discount bond prices.

From Table 5.3 what is important to note is the value of the bond price at time $T = 0$, which has a value of 1, i.e $\hat{v_1}(T_0) = 1$. This bond price is contracted from $[0, 0.5]$ and would have matured and paid an amount of \$1.

Step 4-Evaluating numeraire based caplet payoffs.

Now given the matrix of the forward LIBOR rates given in Table 5.1 , we are interested in the main diagonal elements , as they will be used to determine the future expected caplet payoffs as shown in equation (5.10), i.e. $L_i(T_i)$ $i = 1, 2, ..., N-1$. The table belows shows the main diagonal elements of the forward LIBOR rates "before last settlement" date and on the "last" settlement date.

Time (T_i)	LIBOR rate ($\hat{L_i}(T_i)$)
0	0.05
0.5	0.0491
1	0.0487
1.5	0.0476
2	0.0483
2.5	0.0474
3	0.0477
3.5	0.0464
4	0.0453
4.5	0.0449

Table 5.4: Forward LIBOR rates.

Once we have the forward LIBOR rates shown in Table 5.4 , and the bond prices that have not matured as shown in Table 5.3, we are now in a position to evaluate the caplet prices using Monte Carlo methods. This is done by using equation (5.10). We ran the simulation 10 000 times, and estimate each caplet value as the sample mean of the 10 000 runs. The table below shows the the current time t prices of caplets generated within the LMM and as described in our base scenario.

Time (T_i)	Caplet Price ($\hat{Cpl_i}(t)$)	Standard deviation	95% Confidence interval
0			
0.5	0.9506	0.0170	(0.9395 , 0.9617)
1	0.9260	0.0237	(0.9105 , 0.9415)
1.5	0.9012	0.0284	(0.8826 , 0.9197)
2	0.8764	0.0325	(0.8552 , 0.8977)
2.5	0.8525	0.036	(0.8290 , 0.8761)
3	0.8286	0.0388	(0.8032 , 0.8539)
3.5	0.8042	0.0411	(0.7773 , 0.8310)
4	0.7814	0.0432	(0.7532 , 0.8097)
4.5	0.7596	0.0449	(0.7302 , 0.7889)

Table 5.5: Caplet prices generated within the LMM.

Table 5.5 shows the mean of the i^{th} caplet price for based on the 10000 runs preformed, and its corresponding standard deviation and 95% confidence interval. The value $\hat{Cpl_{0.5}}(t) = 0.9506$, means that the holder of the caplet contract for time period 6 months ($T = 0.5$) to one year ($T = 1$) will pay an amount of \$0.95, to enter into this contract based on a notional principal of \$100. Finally we then compare the caplet prices generated using the Black-Scholes formula for pricing caplets. These prices were generated using a MATLAB script of the Black-Scholes formula (see Appendix 2), where as inputs we used forward LIBOR rates, bond prices and volatility generated in the earlier steps. Below is a table and a graph comparing caplet prices generated using the LMM against those generated using the Black-Scholes formula.

Caplet Prices for a 5 year period ($)			
Time (T_i)	LMM	Black-Scholes	Relative error (%)
0			
0.5	0.9506	0.9317	2.03
1	0.9260	0.9127	1.46
1.5	0.9012	0.8591	4.90
2	0.8764	0.8937	1.94
2.5	0.8525	0.8495	0.35
3	0.8286	0.8646	4.16
3.5	0.8042	0.8010	0.40
4	0.7814	0.7478	4.49
4.5	0.7596	0.7287	4.24

Table 5.6: Caplet prices: LMM vs Black-Scholes.

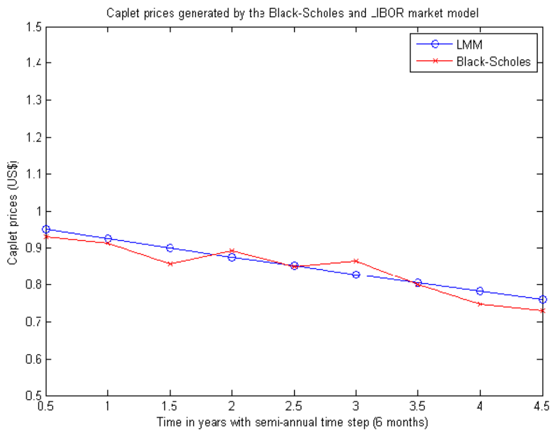

Figure 5.3: Plot of caplet prices LMM vs Black-Scholes

From the results shown in Table 5.6 and Figure 5.3, one can indeed conclude that we indeed get similar caplet prices, using the different models, that is the LIBOR market model (LMM) and the Black-Scholes model. There is an average relative error of 2.66% between the two models, and this is relatively desirable. Lastly the table below shows the prices one would pay to enter an interest cap, which is the a collection or sum of caplets, for our base scenario, based on a $100 notional principal.

	LMM	Black-Scholes
Cap price ($)	7.681	7.589
Standard deviation	0.06572	0.07067
95% Confidence Interval	(7.638,7.723)	(7.543 ,7.635)

Table 5.7: Interest cap price: LMM vs Black-Scholes

From Table 5.7 we can see that the fair price of an interest cap generated using the LMM is relatively similar to the cap price generated by the Black-Scholes formula. Hence we can see that one can efficiently use the LMM to price caplets as an alternative to the more popular Black-Scholes model.

Chapter 6

Conclusion

In this thesis we investigated the LIBOR Market Model (LMM) and Swap Market Model (SMM) for arbitrage-free pricing of interest-rate derivatives such as caplets and swaption respectively. We theoretically construct these models in the continuous-time framework. One can conclude that the dynamics of these "new market models" can be constructed with some mathematical ease, after taking some technical assumptions, as their construction relies on stochastic differential equations (SDE's), unlike in the case of the Black-Scholes model where construction can be based partial differential equations which are more mathematically demanding to implement. Another advantage of the LMM and SMM is that the underlying assets for the models, i.e. forward LIBOR and swap rates, respectively are observable or quoted daily in the financial markets. This makes model implementation much more desirable compared to HJM model (see [11]), where underlying assets are instantaneous interest-rates which are not quoted daily on the market. In this thesis we have shown that even though the Black-Scholes model is considered the standard pricing model for interest-rate derivatives, one can make use of alternative pricing models such as the LMM and SMM to price derivatives. We have shown theoretically that the pricing formula for caplets within the LMM will have similar structure to that of Black-Scholes pricing formula for caplets. Similarly we showed that the SMM pricing formula for payer swaptions is similar to the Black-Scholes pricing formula for swaptions, in structure. This is a result of the fact that we assume the our forward rates, i.e. forward LIBOR rate for the LMM and forward swap rates for the SMM, follow geometric Brownian motion under empirical probability. This implies that the forward rates are log-normally distributed. This is the same assumption made for the Black-Scholes model. From a numerical point of view we generated caplet prices within the LMM using Monte-Carlo simulation. Using a simple base scenario we showed that caplet prices generated by the LMM will be more or less similar to the exact prices calculated using the Black-Scholes formula. Even though we were not exhaustive in the numerical implementation of the LMM and SMM, it is important for one to perform accurate discretizations and calibration of these models, as this may affect prices of interest rate derivatives considerably.

Future Work

In line with this thesis, there are set of objectives we would like to address in the future. Our main objective will be to investigate, understand and elaborate more on the co-terminal measure dynamics of the Swap Market Model, which were developed by Jamshidan ([12]), which was briefly discussed in chapter 4, section 4.2. This is of great interest as it will allow one to understand the discretization process of the SMM for simulation purposes. As a result in the future we would like to perform Euler discretization to the SMM and then simulate swaption prices within the SMM. Once that has been achieved, we will be in position to confirm if the swaption prices generated will be similar to swaption prices generated using the Black-Scholes model. Secondly we would like to investigate simulation of forward swap rates within the LMM, which we theoretically highlighted in chapter 5, subsection 5.2.2. We then intend to generate swaption prices within the LMM. Even though the LMM and SMM are not consistent with each other, Glasserman et.al ([9]) and Kawai ([13]) , suggest some techniques for simulating forward swap rates within the LMM. One can then investigate which models performs better when it comes to pricing swaptions.

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